VI.4. Phragmén-Lindelöf Theorem

Note. Recall that Liouville's Theorem (Theorem IV.3.4) states that a bounded entire function is constant. It is possible to weaken the hypotheses and maintain the conclusion.

Theorem VI.4.A. Suppose f is an entire function, $M > 0$, and $0 < \alpha < 1$. Suppose $|f(z)| \leq M + |z|^\alpha$ for all $z \in \mathbb{C}$. Then f is constant.

Note. The Phragmén-Lindelöf Theorem extends the Maximum Modulus Theorem—Third Version (Theorem VI.1.4) in a manner analogous to the way the above theorem extends Liouville's Theorem. The condition of asymptotic boundedness on the extended boundary of set G is replaced by a condition concerning asymptotic boundedness of function f times another analytic function. Explicitly, we have the following.

Theorem VI.4.1. Phragmén-Lindelöf Theorem.
Let G be a simply connected region and let f be an analytic function on G. Suppose there is an analytic function $\phi : G \to \mathbb{C}$ which is nonzero and is bounded on G. If M is a constant and $\partial_\infty G = A \cup B$ such that

(a) for every $a \in A$ we have $\limsup_{z \to a} |f(z)| \leq M$, and

(b) for every $b \in B$ and $\eta > 0$, we have $\limsup_{z \to b} |f(z)||\phi(z)|^\eta \leq M$,

then $|f(z)| \leq M$ for all $z \in G$.

Note. What the Phragmén-Lindelöf Theorem implies is that condition (a) (which is the hypothesis of the Maximum Modulus Theorem—Third Version) can be replaced with condition (b). Now for the proof.

Note. The following two corollaries are similar to the Phragmén-Lindelöf Theorem (and hence the Maximum Modulus Theorem) but involve specific regions (sectors).

Corollary VI.4.2. Let $a \geq 1/2$ and let $G = \{z \mid |\arg(z)| < \pi/(2a)\}$. Suppose that f is analytic on G and suppose there is a constant M such that $\limsup_{z\to w} |f(z)| \leq M$ for all $w \in \partial G$. If there are positive constants P and $b < a$ such that $|f(z)| \leq P \exp(|z|^b)$ for all z with $|z|$ sufficiently large, then $|f(z)| \leq M$ for all $z \in G$.

Corollary VI.4.4. Let $a \geq 1/2$ and let $G = \{z \mid \arg(z) < \pi/(2a)\}$, and suppose that for every $w \in \partial G$, $\limsup_{z\to w} |f(z)| \leq M$. Moreover, assume that for every $\delta > 0$ there is a constant P (which may depend on δ) such that $|f(z)| \leq P \exp(\delta|z|^a)$ for $z \in G$ and $|z|$ sufficiently large. Then $|f(z)| \leq M$ for all $z \in G$.

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