

# Chapter VII. Compactness and Convergence in the Space of Analytic Functions

**Note.** In this chapter we form a metric space out of the set of all analytic functions on a given region  $G$ . We prove the Riemann Mapping Theorem and introduce both the Gamma Function and the Riemann Zeta Function.

## VII.1. The Space of Continuous Functions $C(G, \Omega)$

**Note.** In this section, we start with a metric space  $(\Omega, d)$  and an open set  $G \subset \mathbb{C}$ . We then consider the set of all continuous functions from  $G$  to  $\Omega$  and show that this set forms a complete metric space. Other topological and analytic topics are explored.

**Definition VII.1.1.** If  $G$  is an open set in  $\mathbb{C}$  and  $(\Omega, d)$  is a complete metric space then denote by  $C(G, \Omega)$  the *set of all continuous functions from  $G$  to  $\Omega$* .

**Note.**  $C(G, \Omega)$  is nonempty since it contains the constant functions. We are primarily interested in the cases where the metric space  $(\Omega, d)$  is either  $(\mathbb{C}, |\cdot|)$  or  $(\mathbb{C}_\infty, d)$  where  $d$  is as defined on page 9 of the text. Of course  $C(G, \mathbb{C})$  contains all analytic functions on  $G$ . Also,  $C(G, \mathbb{C}_\infty)$  contains all meromorphic functions on  $G$  (as shown by Exercise V.3.4).

**Theorem VII.1.2.** If  $G$  is open in  $\mathbb{C}$  then there is a sequence  $\{K_n\}$  of compact subsets of  $G$  such that  $G = \bigcup_{n=1}^{\infty} K_n$ . moreover, the sets  $K_n$  can be chosen to satisfy the following conditions:

- (a)  $K_n \subset \text{int}(K_{n+1})$ ;
- (b)  $K \subset G$  and  $K$  compact imply  $K \subset K_n$  for some  $n$ ;
- (c) Every component of  $\mathbb{C}_{\infty} \setminus K_n$  contains a component of  $\mathbb{C}_{\infty} \setminus G$ .

**Definition.** For  $G$  open in  $\mathbb{C}$ ,  $G = \bigcup_{n=1}^{\infty} K_n$  for compact  $K_n$  and  $K_n \subset \text{int}(K_{n+1})$  as given in Proposition 1.2, define

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\}$$

for all  $f, g \in C(G, \Omega)$ . Also define

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

(We will see below in Corollary VII.1.11 that the topological properties determined by  $\rho$  are independent of the choice of the compact sets  $K_n$ .)

**Note.** The series given in  $\rho(f, g)$  is a positive term series dominated by the geometric series with ration  $1/2$ , so the series converges (by the Direct Comparison Test) and so  $\rho(f, g)$  is defined for all  $f, g \in C(G, \Omega)$ . We now show that  $\rho$  is a metric on  $C(G, \Omega)$ .

**Lemma VII.1.5.** If  $(S, d)$  is a metric space then

$$\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$$

is also a metric on  $S$ . A set is open in  $(S, d)$  if and only if it is open in  $(S, \mu)$ ; a sequence is a Cauchy sequence in  $(S, d)$  if and only if it is a Cauchy sequence in  $(S, \mu)$ .

**Proof.** This is done in Exercise VII.1.1. ■

**Proposition VII.1.6.**  $(C(G, \Omega), \rho)$  is a metric space.

**Note.** The following three results relate open sets and compact sets in metric space  $(C(G, \Omega), \rho)$ . Conway states: “These who know the appropriate definition will recognize that [the following] lemma says that two uniformities are equivalent.”

**Lemma VII.1.7.** Let the metric  $\rho$  be defined as above:

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}$$

for  $f, g \in C(G, \Omega)$  where  $G = \cup_{n=1}^{\infty} K_n$  for compact  $K_n$  with  $K_n \subset \text{int}(K_{n+1})$  and  $\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\}$ . If  $\varepsilon > 0$  is given then there is  $\delta > 0$  and a compact set  $K \subset G$  such that for  $f, g \in C(G, \Omega)$ ,

$$\sup\{d(f(z), g(z)) \mid z \in K\} < \delta \implies \rho(f, g) < \varepsilon.$$

Conversely, if  $\delta > 0$  and a compact set  $K$  are given then there is  $\varepsilon > 0$  such that for  $f, g \in C(G, \Omega)$ ,

$$\rho(f, g) < \varepsilon \implies \sup\{d(f(z), g(z)) \mid z \in K\} < \delta.$$

**Proposition VII.1.10.**

- (a) A set  $\mathcal{O} \subset (C(G, \Omega), \rho)$  is open if and only if for each  $f \in \mathcal{O}$  there is a compact set  $K$  and a  $\delta > 0$  such that  $\mathcal{O} \supset \{g \mid d(f(z), g(z)) < \delta \text{ for } z \in K\}$ .
- (b) A sequence  $\{f_n\}$  in  $(C(G, \Omega), \rho)$  converges to  $f$  if and only if  $\{f_n\}$  converges to  $f$  uniformly on all compact subsets of  $G$ .

**Corollary VII.1.11.** The collection of open sets is independent of the choice of the sets  $\{K_n\}$ . That is, if  $G = \cup_{n=1}^{\infty} K'_n$  where each  $K'_n$  is compact and  $K'_n \subset \text{int}(K_{n+1})$  and if  $\mu$  is the metric defined by the sets  $\{K'_n\}$  then a set is open in  $(C(G, \Omega), \mu)$  if and only if it is open in  $(C(G, \Omega), \rho)$ .

**Proof.** This follows from Proposition 1.10 in which open sets in  $(C(G, \Omega), \rho)$  are classified in terms of compact sets, but with no specific appeal to the  $\{K_n\}$  or  $\{K'_n\}$ . ■

**Note.** Corollary VII.1.11 tells us that the topology on  $C(G, \Omega)$  is the same regardless of how the compact sets  $K_n$  are chosen. In a normed linear space, two norms are said to be “equivalent” when they induce the same topologies (see my online notes for Fundamentals of Functional Analysis [MATH 5740] on the Section “Comparison of Norms”: <http://faculty.etsu.edu/gardnerr/Func/notes/2-6.pdf>). This would inspire us to call any two metrics on  $C(G, \Omega)$  produced by a sequence of compact sets as described above as “equivalent” (though this does not mean that the metrics give the distances between corresponding points).

**Note.** Conway claims that the above results of this section still hold even when we drop the condition  $K_n \subset \text{int}(K_{n+1})$  (see page 145). However, to establish this requires “some extra effort” such as the Baire Category Theorem. This result deals with subsets of a metric space. A subset  $E$  of a metric space  $X$  is of the *first category* if  $E$  is the union of a countable collection of nowhere dense subsets of  $X$ . A set that is not of the first category is of the *second category*. The Baire Category Theorem states that an open subset of a complete metric space is of the second category. See page 214 of Royden and Fitzpatrick’s *Real Analysis*, 4th Edition (Prentice Hall, 2010) for these definitions.

**Note.** We have not yet addressed completeness. Throughout the rest of this section we assume that metric space  $(\Omega, d)$  is complete. This implies that  $C(G, \Omega)$  is complete, as we now see.

**Proposition VII.1.12.** If metric space  $(\Omega, d)$  is complete, then metric space  $C(G, \Omega)$  is complete.

**Note.** The remainder of this section is not needed for the rest of this chapter. “The next definition is derived from the classical origins of the subject” (Conway, page 146). We use the definition in the statement and proof of the Arzela-Ascoli Theorem which “is a deep result which has proved extremely useful in many areas of analysis” (Conway, page 146). Much of the rest of this section is in Ahlfors’s *Complex Analysis*, Third Edition (McGraw-Hill, 1979) in Sec-

tion 5.5. The Arzela-Ascoli Theorem is covered in Section 10.1 of Royden and Fitzpatrick's *Real Analysis* in a chapter on metric spaces. It is in Section 9.2 of Promislow's *A First Course in Functional Analysis* (Wiley & Sons, 2008)—this is the text used in our Introduction to Functional Analysis class (MATH 5740). See <http://faculty.etsu.edu/gardnerr/Func/notes.htm> for more details on the functional analysis material.

**Definition VII.1.14.** A set  $\mathcal{F} \subset C(G, \Omega)$  is *normal* if each sequence in  $\mathcal{F}$  has a subsequence which converges to a function  $f$  in  $C(G, \Omega)$ .

**Note.** The following result makes the structure of normal sets a little more tangible. The proof of the result is “left to the reader.”

**Proposition VII.1.15.** A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if its closure is compact.

**Proposition VII.1.16.** A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if for every compact set  $K \subset G$  and every  $\delta > 0$ , there are functions  $f_1, f_2, \dots, f_n \in \mathcal{F}$  such that for  $f \in \mathcal{F}$  there is at least one  $k$ ,  $1 \leq k \leq n$ , with

$$\sup\{d(f(z), f_k(z)) \mid z \in K\} < \delta.$$

**Definition.** Let  $(X_n, d_n)$  be the metric space for each  $n \in \mathbb{N}$  and let  $X = \prod_{n=1}^{\infty} X_n$  (the Cartesian product). For  $\xi = \{x_n\}$  and  $\eta = \{y_n\}$  in  $X$  (so  $x_n \in X_n$  and  $y_n \in X_n$ ) define

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

**Proposition VII.1.18.** The space  $(\prod_{n=1}^{\infty} X_n, d)$  of the previous definition is a metric space. If  $\xi^k = \{x_n^k\}_{n=1}^{\infty}$  is in  $X = \prod_{n=1}^{\infty} X_n$  then  $\xi^k \rightarrow \xi = \{x_n\}$  if and only if  $x_n^k \rightarrow x_n$  for all  $n \in \mathbb{N}$ . also, if each  $(X_n, d)$  is compact then  $X$  is compact.

**Note.** The following definition “plays a central role in the Arzela-Ascoli Theorem.”

**Definition VII.1.21.** A set  $\mathcal{F} \subset C(G, \Omega)$  is *equicontinuous at a point*  $z_0 \in G$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that for  $|z - z_0| < \delta$ ,  $d(f(z), f(z_0)) < \varepsilon$  for every  $f \in \mathcal{F}$ . Set  $\mathcal{F}$  is *equicontinuous over a set*  $E \subset G$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that for  $z, z' \in E$  and  $|z - z'| < \delta$  we have  $d(f(z), f(z')) < \varepsilon$  for all  $f \in \mathcal{F}$ .

**Note.** If  $\mathcal{F} = \{f\}$  then “ $\mathcal{F}$  is equicontinuous at  $z_0$ ” simply means that  $f$  is continuous at  $z_0$ . In this case, “ $\mathcal{F}$  is equicontinuous on set  $E$ ” means that  $f$  is uniformly continuous on  $E$ . For larger sets  $\mathcal{F}$ , Conway describes equicontinuity on a set as “uniform uniform continuity.”

**Proposition VII.1.22.** Suppose  $\mathcal{F} \subset C(G, \Omega)$  is equicontinuous at each point of  $G$ . Then  $\mathcal{F}$  is equicontinuous over each compact subset of  $G$ .

**Note.** Now for the Arzela-Ascoli Theorem which classifies normal sets  $\mathcal{F} \subset C(G, \Omega)$  and relates normality to equicontinuity.

**Theorem VII.1.23. Arzela-Ascoli Theorem**

A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if the following two conditions are satisfied:

- (a) For each  $z \in G$ , we have that  $\{f(z) \mid f \in \mathcal{F}\}$  has compact closure in  $\Omega$ ;
- (b)  $\mathcal{F}$  is equicontinuous at each point of  $G$ .

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