VII.2. Spaces of Analytic Functions

Note. In this section, we consider the subset of $C(G, \mathbb{C})$ consisting of analytic functions on $G \subset \mathbb{C}$. We show this set is also a complete metric space. We also classify a normal set in this space.

Definition. Let G be an open subset of $\mathbb{C}$. We denote the collection of all analytic functions in $C(G, \mathbb{C})$ as $H(G)$.

Note. The symbol "H" is used because another (older) term for analytic is "holomorphic." Another old term for analytic is "regular." The symbol $A(G)$ denotes the set of all complex valued functions continuous on G^- and analytic on G.

Note. We treat $H(G)$ as a metric space with the same metric ρ as $C(G, \mathbb{C})$. As in Section VII.1, we define ρ by introducing compact sets K_n for $n \in \mathbb{N}$ such that $G = \cup_{n=1}^{\infty} K_n$ and $K_n \subset \text{int}(K_{n+1})$ (such sets exist by Proposition VII.1.2). We define

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\} = \max\{|f(z) - g(z)| \mid z \in K_n\},$$

since the metric on $\mathbb{C}$ is the metric induced by modulus and the fact that analytic functions are continuous on the compact sets K_n. We then have

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

Notice that if a sequence of functions $\{f_n\} \subset C(G, \mathbb{C})$ converges uniformly on G to f then the sequence also converges to f with respect to metric ρ.

Theorem VII.2.1. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G, \mathbb{C})$ such that $\lim_{n\to\infty} f_n = f$, then f is analytic and the derivatives satisfy $\lim_{n\to\infty} f_n^{(k)} = f^{(k)}$ for each $k \in \mathbb{N}$.

Note. Theorem VII.2.1 shows that for any sequence $\{f_n\} \subset H(G)$ such that $\{f_n\} \to f$ in $C(G, \mathbb{C})$, we have $f \in H(G)$. Since $C(G, \mathbb{C})$ is complete by Proposition VII.1.12, we see that any Cauchy sequence in $H(G)$ converges in $H(G)$. This gives the following.

Corollary VII.2.3. The set of all analytic functions on region G, $H(G)$, is a metric space under metric ρ.

Note. By Proposition VII.1.10(b), if a sequence of functions converges uniformly on all compact subsets of $\mathbb{C}$, then the sequence also converges in $C(G, \mathbb{C})$. This, combined with the observation that $f_n \to f$ in $C(G, \mathbb{C})$ implies that $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$ given in Theorem VII.2.1 gives the following.

Corollary VII.2.4. If $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note. Theorem VII.2.1 does not hold in the real setting. For example, the absolute value function is a uniform limit of continuous functions. Another example is given be $f_n(x) = x^n/n$ on $[0,1]$. The sequence uniformly converges to 0, but $\{f_n'(x)\} = \{x^{n-1}\}$ does not converge uniformly on $[0,1]$ and this (pointwise) limit is not continuous. At this stage, Conway declares analytic functions "special" and supports this claim with the following.

Theorem VII.2.5. Hurwitz's Theorem. Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a;R) \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a;R)$.

Corollary VII.2.6. If G is a region and $\{f_n\} \subset H(G)$ converges to f in $H(G)$ and each f_n never vanishes on G then either $f \equiv 0$ or f never vanishes.

Note. We need another definition to classify normal families in $H(G)$.

Definition VII.2.7. A set $\mathcal{F} \subset H(G)$ is *locally bounded* if for each point $a \in G$ there are constants $M > 0$ and $r > 0$ such that for all $f \in \mathcal{F}$ we have $|f(z)| \leq M$ for $|z-a| < r$.

Note. We can equally well say that $\mathcal{F}$ is locally bounded if there is $r > 0$ such that

$$\sup\{|f(z)| \mid |z - q| < r, f \in \mathcal{F}\} < \infty.$$

That is, $\mathcal{F}$ is locally bounded if for each $a \in G$ there is a disk containing a on which $\mathcal{F}$ is uniformly bounded.

Lemma VII.2.8. A set $\mathcal{F} \subset H(G)$ is locally bounded if and only if for each compact set $K \subset G$ there is a constant $M > 0$ such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and for all $z \in K$.

Note. We now classify normal subsets of $H(G)$.

Theorem VII.2.9. Montel's Theorem. A family $\mathcal{F} \subset H(G)$ if normal if and only if $\mathcal{F}$ is locally bounded.

Note. We can now classify compact subsets of $H(G)$ in a way reminiscent of the Heine-Borel Theorem.

Corollary VII.2.10. A set $\mathcal{F} \subset H(G)$ is compact if and only if it is closed and locally bounded.

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