VII.3. Spaces of Meromorphic Functions

Note. Function f on region G is meromorphic if it is analytic on G except for poles. Isolated singularity $z = a$ of f is a pole if $\lim_{z\to a} |f(z)| = \infty$. If we define $f(z) = \infty$ at each pole of meromorphic function f, then by Exercise V.3.4, $f : G \to \mathbb{C}_\infty$ is continuous where the metric d on $\mathbb{C}_\infty$ is as given in Section I.6. The Extended Plane and Its Spherical Representation:

$$\begin{aligned} d(z_1, z_2) &= \frac{2|z_1 - z_2|}{\{(1 + |z_1|^2)(1 + |z_2|^2\}^{1/2}} \text{ for } z_1, z_2 \in \mathbb{C} \\ d(z, \infty) &= \frac{2}{(1 + |z|^2)^{1/2}} \text{ for } z \in \mathbb{C}. \end{aligned}$$

Note. In this section we consider $M(G)$, the set of all meromorphic functions on region $G \subset \mathbb{C}$ and treat $M(G)$ as a subset of $C(G, \mathbb{C}_\infty)$. So the metric on $M(G)$ is

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}$$

where $G = \cup_{n=1}^{\infty} K_n$, each K_n is compact, $K_n \subset \text{int}(K_{n+1})$, and

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\}.$$

Note. For nonzero $z_1, z_2 \in \mathbb{C}$, the metric in $\mathbb{C}_\infty$ satisfies

$$\begin{aligned} d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) &= \frac{2|1/z_1 - 1/z_2|}{\{(1 + |1/z_1|^2)(1 + |1/z_2|^2)\}^{1/2}} \times \frac{|z_1||z_2|}{|z_1||z_2|} \\ &= \frac{2|z_1 - z_2|}{\{(1 + |z_1|^2)(1 + |z_2|^2)\|^{1/2}} = d(z_1, z_2). \end{aligned}$$

For nonzero $z \in \mathbb{C}$,

$$d\left(\frac{1}{z}, \infty\right) = \frac{2}{\{z + |1/z|^2\}^{1/2}} = \frac{2|z|^2}{\{1 + |z|^2\}^{1/2}} = d(z, 0).$$

Note. Exercise II.3.4 states: Let $a_n, z \in \mathbb{C}$ and let d be the metric on $\mathbb{C}_\infty$. Then $|z_n - z| \to 0$ if and only if $d(z_n, z) \to 0$. Also, if $|z_n| \to \infty$ then $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}_\infty$.

Note. The following result compares the metric spaces $\mathbb{C}$ under the normal metric on $\mathbb{C}$, and $\mathbb{C}_\infty$ under the metric d. We use the notation "$B_\infty(a; r)$" to indicate a ball in $\mathbb{C}_\infty$ with center a and radius r. The proof is to be given in Exercise VII.3.1.

Proposition VII.3.3.

(a) If $a \in \mathbb{C}$ and $r > 0$, then there is $\rho > 0$ such that $B_\infty(a; \rho) \subset B(a; r)$.

(b) Conversely, if $\rho > 0$ is given and $a \in \mathbb{C}$ then there is a number $r > 0$ such that $B(a : r) \subset B_\infty(a; \rho)$.

(c) If $\rho > 0$ is given then there is a compact set $K \subset \mathbb{C}$ such that $\mathbb{C}_\infty \setminus K \subset B_\infty(\infty; \rho)$.

(d) Conversely, if a compact set $K \subset \mathbb{C}$ is given, then there is $\rho > 0$ such that $B_\infty(\infty, \rho) \subset \mathbb{C}_\infty \setminus K$.

Note. It is easy to see that $M(G)$ is not complete. Consider $\{f_n\}$ where $f_n(z) = n$. The sequence $\{f_n\}$ converges to the constant function $f(z) = \infty$. Notice that f is not meromorphic since it is not analytic on G except for (isolated) poles, but $f \in C(G, \mathbb{C}_\infty)$. We will see below in Corollary VII.3.5 that $M(G) \cup \{f\}$, where $f(z) = \infty$ for all $z \in \mathbb{C}$ (denoted $f \equiv \infty$), is complete. We first prove a more general result.

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic or $f \equiv \infty$.

Note. By Proposition VII.1.12, $C(G, \Omega)$ is complete where G is an open subset of $\mathbb{C}$ and (Ω, d) is a complete metric space. So any Cauchy sequence $\{f_n\} \subset C(G, \mathbb{C}_\infty)$ is convergent in $C(G, \mathbb{C}_\infty)$. So if $\{f_n\}$ is a Cauchy sequence in $M(G) \subset C(G, \mathbb{C})$ then $f_n \to f$ for some $f \in C(G, \mathbb{C}_\infty)$ and by Theorem VII.3.4 we have that either $f \in M(G)$ or $f \equiv \infty$. This gives the following.

Corollary VII.3.5. Let G be a region in $\mathbb{C}$ (i.e., an open connected subset in $\mathbb{C}$). Then the meromorphic functions on G combined with $f \equiv \infty$ on G, $M(G) \cup \{\infty\}$, is a complete metric space under metric ρ.

Note. If $\{f_n\}$ is a convergent sequence in $H(G) \subset C(G, \mathbb{C}_\infty)$ then by Theorem VII.1.12, $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. By the second claim in Theorem VII.3.4, either $f \equiv \infty$ or f is analytic. this gives the following.

Corollary VII.3.6. Let G be a region in $\mathbb{C}$. Then the analytic functions on G combined with $f \equiv \infty$, $H(G) \cup \{\infty\}$, is a closed subset of $C(G, \mathbb{C}_\infty)$.

Note. We already know that $H(G)$ is complete by Corollary VII.2.3. We now concentrate on $M(G)$ and its normal subsets. Recall that a set $\mathcal{F}$ (in $C(G,\Omega)$) is normal if each sequence in $\mathcal{F}$ has a subsequence which converges.

Note. Recall that $\mathcal{F} \subset C(F,\Omega)$ is normal if and only if its closure is compact (Proposition VII.1.15). Montel's Theorem states that $\mathcal{F} \subset H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded. To explore normality in $M(G)$, we need to consider the quantity $2|f'(z)|/(1+|f(z)|^2)$, as we'll see in Theorem VII.3.8. However, this quantity is not defined where f has a pole since the derivative is undefined there. So we'll define the quantity at poles of f using limits. Recall that if f has an isolated pole of order $m \geq 1$ at $z = a$ then

$$f(z) = g(z) + \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{z-a}$$

for z in some disk about a and g analytic in the disk (see Section V.1; equation (V.1.7) in the text). So for $z \neq 1$,

$$f'(z) = g(z) - \left[\frac{mA_m}{(z-a)^{m+1}} + \frac{(m-1)A_{m-1}}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)^2}\right].$$

Thus,

$$\frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2\left|\frac{mA_m}{(z-a)^{m+1}} + \frac{(m-1)A_{m-1}}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)^2} - g'(z)\right|}{1+\left|\frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g'(z)\right|^2}$$

$$= \frac{2|z-a|^{m-1}\left|mA_m + (m-1)A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}\right|}{|a-z|^{2m} + |A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + g(z)(z-a)^m|^2}$$

(multiplying by $|z-a|^{2m}/|z-a|^{2m}$). So if $m \geq 2$ then $\lim_{z\to a} \frac{2|f'(z)|}{1+|f(z)|^2} = 0$ and if $m = 1$ then

$$\lim_{z\to a} \frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2m|A_m|}{|A_m|^2} = \frac{2}{|A_1|}.$$

Definition. If f is a meromorphic function on the region G then define $\mu(f) : G \to \mathbb{R}$ by $\mu(f)(z) = \dfrac{2|f'(z)|}{1+|f(z)|^2}$ for z not a pole of f and

$$\mu(f)(a) = \mu(f)(a) = \begin{cases} 0 & \text{if } m > 2 \\ 2/|A_1| & \text{if } m = 1 \end{cases}$$

when a is a pole of f of order m (where A_1 is as defined in the above note).

Note. $\mu(f) : G \to \mathbb{R}$ defined above is continuous on G (clearly at nonpoles and by definition at poles) so $\mu(f) \in C(G, \mathbb{R})$.

Note. Recall that $\mathcal{F} \subset H(G)$ is normal if and only it it is locally bounded and "locally bounded" means that for each compact set $K \subset G$ there is a constant M such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and $z \in K$. So we see, at least in $H(G)$, that normality is related to a type of uniform boundedness. Conway informally inspires the introduction of $\mu(f)$ as follows. If $f : G \to \mathbb{C}_\infty$ is meromorphic then for z "close" to z' (not poles of f) we have

$$d(f(z), f(z')) = \frac{2|f(z) = f(z')|}{\{(1+|f(z)|^2)(z+|f(z')|^2)\}^{1/2}}$$
$$= \frac{2|z-z'||f(z)-f(z')|/|z-z'|}{\{(1+|f(z)|^2)(1+|f(z')|^2)\}^{1/2}} \approx \frac{2|z-z'||f'(z)|}{1+|f(z)|^2} = \mu(f)(z)|z-z'|.$$

So if $\mu(f)$ is bounded, say $\mu(f) \leq M$, then $d(f(z), f(z')) \leq M|z-z'|$ and so f is Lipschitz. So if for set $\mathcal{F} \subset M(G)$, $\mu(f)$ is uniformly bounded on $\mathcal{F}$, then $\mathcal{F}$ is a "uniformly Lipschitz" set of functions. In Exercise VII.2.4, an alternative proof of Montel's Theorem is outlined in which it is shown that for $\mathcal{F} \subset H(G)$:

$$\text{locally bounded} \Longrightarrow \text{locally Lipschitz} \Longrightarrow \text{equicontinuous}.$$

The Arzela-Ascoli Theorem (Theorem VII.1.23) then gives normality. So here (in $M(G)$), we will use local boundedness of $\mu(f)$ on $\mathcal{F}$ to get locally Lipschitz on $\mathcal{F}$ and equicontinuity, in the following.

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Note. Theorem VII.3.8. refers to $\mathcal{F} \subset M(G)$ as being normal in $C(G, \mathbb{C}_\infty)$, not in $M(G)$! This is because $M(G)$ is not complete (but $M(G) \cup \{\infty\}$ is, by Corollary VII.3.5). For example, with $f_n(z) = nz$, $n \in \mathbb{N}$, then $\mathcal{F} = \{f_n\}$ is normal in $C(G, \mathbb{C}_\infty)$ since $f_n \to f \equiv \infty$ (and so every subsequence converges in $C(G, \mathbb{C}_\infty)$. Also, $\mu(f_n)(z) = \dfrac{2n}{1 + n^2|z|^2}$ is locally bounded (recall that local boundedness concerns a given compact subset of G). But $\mathcal{F}$ is not normal in $M(G)$ since $\{f_n\}$ and any subsequence of $\{f_n\}$ converges to $f \equiv \infty$, but $f \equiv \infty \notin M(G)$.

Note. We will get a classification of meromorphic functions on G in Section VII.5 (The Factorization Theorem). If $f \in M(G)$, then there are $g, h \in H(G)$ such that $f = g/h$ on G (Corollary VII.5.20); that is, meromorphic functions are quotients of analytic functions).

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