VII.4. The Riemann Mapping Theorem

Note. In this section, we use several of the earlier results in this chapter, but change our focus from spaces of functions to simply connected regions. We prove that for any two proper simply connected regions in $\mathbb{C}$ (i.e., any two simply connected open sets not equal to $\mathbb{C}$), there is an analytic function mapping one region one to one and onto the other region. This is the Riemann Mapping Theorem.

Definition VII.4.1. A region G_1 if *conformally equivalent* to a region G_2 if there is analytic function $f : G_1 \to \mathbb{C}$ such that f is one to one and $f(G_1) - G_2$ (that is, f is onto G_2).

Note. Conformal equivalence is an equivalence relation:

(1) Reflexive: For all regions G, G is conformally equivalent to itself as shown by considering the identity function;

(2) Symmetric: If G_1 is conformally equivalent to G_2 under function $f : G \to G_2$, then $f^{-1} : G_2 \to G_1$ is analytic by Corollary IV.7.6, so G_2 is conformally euqivalent to G_1;

(3) Transitive: If G_1 is conformally equivalent to G_2 under analytic f, and G_2 is conformally euivalent to G_3 under analytic f_2 then $f_c \circ f_1 : G_1 \to G_3$ is analytic by the Chain Rule (Theorem III.2.4) and so G_1 is conformally equivalent to G_3.

Note. The entire complex plane $\mathbb{C}$ cannot be conformally equivalent to a bounded set. If it were, then there would be analytic $f : \mathbb{C} \to B$ where B is some bounded set. But then f is bounded on $\mathbb{C}$ and so by Liouville's Theorem (Theorem IV.3.4) f must be constant, contradicting the requirement that f is one to one.

Lemma VII.4.A. If G_1 is simply connected and G_1 is conformally equivalent to G_2 then G_2 is simply connected.

Note. We now state the Riemann Mapping Theorem which, in the verbiage of "conformally equivalent," states that every simply connected region in $\mathbb{C}$, which is not all of $\mathbb{C}$, is conformally equivalent to the disk $|z| < 1$ (and so any to such regions are conformally equivalent to each other).

Theorem VII.4.2. The Riemann Mapping Theorem.
Let G be a simply connected region which is not the whole plane $\mathbb{C}$ and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one to one; and

(c) $f(G) = \{z \mid |z| < 1\}$.

Note. The proof will require a lemma, As we'll see, the only property of a simply connected region which is needed is that every nonvanishing analytic function on the region has an analytic square root.

Lemma VII.4.3. Let G be a region which is not the whole plane and such that every nonvanishing analytic function on G has an analytic square root. If $a \in G$ then there is an analytic function f on G such that:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one to one; and

(c) $f(G) = \{z \mid |z| < 1\}$.

Note. We are now ready for the proof of the Riemann Mapping Theorem.

Corollary VII.4.10. Among the simply connected regions, there are only two equivalence classes with respect to conformally equivalence; one class consisting of $\mathbb{C}$ alone and the other class containing all the proper simply connected regions.

Note. A number of conditions equivalent to a region being simply connected will be given in Section VIII.2, "Simple Connectedness" (see Theorem VIII.2.2). The conditions involve topological, analytic, and algebraic conditions.

Revised: 8/6/2017