VII.5. The Weierstrass Factorization Theorem

Note. Conway motivates this section with the following question: "Given a sequence $\{a_k\}$ in G which has no limit point on G and a sequence of [positive] integers $\{m_k\}$, is there a function f which is analytic on G and such that the only zeros of f are at the points a_k, with the multiplicity of the zero at a_k equal to m_k?" If the set $\{a_k\}$ is finite, we simply take f as the polynomial $f(z) = (z - a_1)^{m_1}(z - a_2)^{m_2} \cdots (z - a_n)^{m_n}$. If the set $\{a_k\}$ is infinite (such as is the case for a sine function, for example), we need to discuss infinite products of complex numbers. By the way, the answer to the question is yes, as seen in Theorem VII.5.15.

Definition VII.5.1. If $\{z_n\}$ is a sequence of complex numbers and if $z = \lim_{n\to\infty} \left(\prod_{k=1}^{n} z_k \right)$ exists, then z is the *infinite product* of the numbers z_n and is denoted $z = \prod_{n=1}^{\infty} z_n$.

Note. If any $z_n = 0$ then $\prod_{n=1}^{\infty} z_n = 0$.

Lemma VII.5.A. Let $\{z_n\}$ be a sequence of nonzero complex numbers. Suppose $\prod_{k=1}^{\infty} z_k$ exists. If $\prod_{k=1}^{\infty} a_k \neq 0$ then $\lim_{n\to\infty} z_n = 1$.

Note. Notice that for $z_n = a$ for $n \in \mathbb{N}$ where $|a| < 1$, we have $\lim_{n\to\infty} z_n = a \neq 0$ yet $\prod_{k=1}^{\infty} z_k = 0$.

Note. Since the logarithm of a product is the sum of the logarithms of the factors of the product, we will often deal with the convergence of an infinite product in terms of the convergence of the *series* of associated logarithms. Now with $p_n = \prod_{k=1}^n z_k$ we have $\log(p_n) = \log\left(\prod_{k=1}^n z_k\right) = \sum_{k=1}^n \log z_k$, provided "log" represents a branch of the logarithm on which each z_k is defined. If $\prod_{K=1}^\infty z_k$ exists and is not zero then by Lemma VII.5.A, $\lim_{n\to\infty} z_n = 1$, so "eventually" $\text{Re}(z_n) > 0$ and we can use the principal branch of the logarithm. With $s_n = \log p_n = \sum_{k=1}^n \log z_k$, we have for $s = \lim_{n\to\infty} s_n$ that $e^s = s^{\lim_{n\to\infty} s_n}$ since e^z is continuous. Also $\exp(s_n) = p_n = \prod_{k=1}^n a_k$ so

$$\prod_{k=1}^\infty a_k = \lim_{n\to\infty} p_n = \lim_{n\to\infty} \exp(s_n) = \exp\left(\lim_{n\to\infty} s_n\right) = e^s \neq 0.$$

So avoiding zero, we can say that $\prod_{n=1}^\infty z_k$ converges if $\sum_{k=1}^\infty \log z_n$ converges. In fact, the converse also holds, as follows.

Proposition VII.5.2. Let $\text{Re}(z) > 0$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^\infty z_n$ converges to a nonzero complex number if and only if the series $\sum_{n=1}^\infty \log z_n$ converges.

Lemma VII.5.B. If $|z| < 1/2$ then $\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|$.

Proposition VII.5.4. Let $\text{Re}(z) > -1$. Then the series $\sum_{n=1}^\infty \log(1+z_n)$ converges absolutely if and only if the series $\sum_{n=1}^\infty z_n$ converges absolutely.

Note. We cannot parallel the definition of absolute convergence of a series in defining the absolute convergence of an infinite product. For example, if $z_1 = (-1)^n$ then $\prod_{n=1}^{\infty} |z_n| = 1$, but $\prod_{n=1}^{\infty} z_n = \prod_{n=1}^{\infty}(-1)^n$ is undefined. So we turn to Proposition VII.5.2 for inspiration.

Definition VII.5.5. If $\text{Re}(z_n) > 0$ for all $n \in \mathbb{N}$, then the infinite product $\prod_{n=1}^{\infty} \log z_n$ converges absolutely.

Lemma VII.5.C. Let $\{z_n\}$ be a sequence of complex numbers with $\text{Re}(z_n) > 0$ for all $n \in \mathbb{N}$ and suppose $\prod_{n=1}^{\infty} z_n$ converges absolutely. Then

(a) $\prod_{n=1}^{\infty} z_n$ converges; and

(b) any rearrangement of $\{z_n\}$, say $\{z_m\}$ (where $m = f(n)$ for some given one to one and onto $f : \mathbb{N} \to \mathbb{N}$) converges absolutely.

Corollary VII.5.6. If $\text{Re}(z_n) > 0$ then the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty}(z_n - 1)$ converges absolutely.

Note. In the following, we turn our attention to sequences of functions and uniform convergence.

Lemma VII.5.7. Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(z) \to f(z)$ uniformly for $x \in X$. If there is a constant a such that $\mathrm{Re}(f(z)) \leq a$ for all $x \in X$, then $\exp(f_n(x)) \to \exp(f(x))$ uniformly for $x \in X$.

Lemma VII.5.8. Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X to $\mathbb{C}$ such that $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely and uniformly for $x \in X$. Then the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for $x \in X$. Also, there is $n_0 \in \mathbb{N}$ such that $f(z) = 0$ if and only if $g_n(x) = -1$ for some n where $1 \leq n \leq n_0$.

Note. Lemmas VII.5.7 and VII.5.8 apply to functions from a set or metric space to $\mathbb{C}$. For the remainder of this section, we concentrate on analytic functions $f : G \to \mathbb{C}$ where G is a region in $\mathbb{C}$.

Theorem VII.5.9. Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ (i.e., a sequence of analytic functions) such that no f_n is identically zero. If $\sum_{n=1}^{\infty}(f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of G, then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the function f_n at a.

Note. In terms of the original question stated at the beginning of this section, to create an analytic function on G with zeros $\{a_n\}$, we try to create functions g_n analytic and nonzero on G such that $\prod_{n=1}^{\infty}(z-a_n)g_n(z)$ is analytic and has zeros only at the points a_n (with multiplicity dealt with by repeating the zeros in sequence $\{a_n\}$ a required number of times). Theorem VII.5.9 implies that this will work if $\sum_{n=1}^{\infty}|(z-a_n)g_n(z)-1|$ converges absolutely on compact subsets of G. So the trick is to find the appropriate $g_n(z)$. This was first accomplished by Weierstrass who introduced "elementary factors" which have a simple zero at a given point and is nonzero elsewhere.

Definition VII.5.10. An *elementary factor* is a function $E_p(z)$ for $p = 0, 1, 2, \ldots$ as follows:

$$\begin{aligned} E_0(z) &= 1 - z, \\ E_p(z) &= (1-z)\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right) \text{ for } p \geq 1. \end{aligned}$$

Note. The function $E_p(z/a)$ has a simple zero at $z = a$ and no other zero. To apply Theorem VII.5.9 to a collection of elementary factors, we need the following lemma. Also, notice that $z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots + \dfrac{z^p}{p}$ is a partial sum for the power series for $-\log(1-z)$: $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ for $|z| < 1$. So for p "large," $z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots + \dfrac{z^p}{p} \approx -\log(1-z)$ and $\exp\left(z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \cdots + \dfrac{z^p}{p}\right) \approx \exp(-\log(1-z)) = \dfrac{1}{1-z}$, so $E_p(z) \approx 1$ for p "large," as expected.

Lemma VII.5.11. If $|z| \leq 1$ and $p \geq 0$ then $|1 - E_p(z)| \leq |z|^{p+1}$.

Note. We first give a solutions to a simplified version of the original question. Instead of considering a region $G \subset \mathbb{C}$, we start with the case $G = \mathbb{C}$.

Theorem VII.5.12. Let $\{a_n\}$ be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} |z_n| = \infty$ and $a_z \neq 0$ for all $n \geq 1$. Suppose that no complex number is repeated in the sequence an infinite number of times. If $\{p_n\}$ is any sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \qquad (5.13)$$

for all $r > 0$, then $f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a)$ converges in $H(\mathbb{C})$ (and so is analytic on $\mathbb{C}$). The function f is an entire function with zeros only at the points a_n If z_0 occurs in the sequence $\{a_n\}$ exactly n times then f has a zero at $z = z_0$ of multiplicity m. Furthermore, if $p_n = n - 1$ then (5.13) will be satisfied.

Note. We can potentially take liberties in the choices of $\{p_n\}$. Conway says that the smaller the p_n the "more elementary" the elementary factor $E_{p_n}(z/a_n)$ (page 170).

Theorem VII.5.14. The Weierstrass Factorization Theorem.
Let f be an entire function and let $\{a_n\}$ be the nonzero zeros of f repeated according to multiplicity. Suppose f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at 0 means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\int} E_{p_n}\left(\frac{z}{a_n}\right).$$

Note. We are now ready to answer the question posed at the beginning of this section.

Theorem VII.5.15. Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit points in G. Let $\{m_j\}$ be a sequence of nonnegative integers. Then there is an analytic function f defined on G whose only zeros are at the points a_j. Furthermore, a_j is a zero of f of multiplicity m_j.

Note. Theorem VII.5.15 allows us to classify meromorphic function on G in terms of quotients of analytic functions on G.

Corollary VII.5.20. If f is a meromorphic function on an open set G then there are analytic functions g and h on G such that $f = g/h$.

Note. When meromorphic f on G is written as $f = g/h$ where g and h are analytic on G, then f and g have the same zeros, and the poles of f correspond to the zeros of h (notice that g and h so not share any zeros).

Note. Conway state (page 173) that "$M(G)$ is the quotient field [he should use the term "field fo quotients" here] of the integral domain $H(G)$." Let's dissect this algebraic language. We refer to class notes for Introduction to Modern Algebra 1 and 2 (MATH 4127/5127, 4137/5137); see `http://faculty.etsu.edu/gardnerr/4127/notes.htm` and `http://faculty.etsu.edu/gardnerr/4127/notes2.htm`.

Definition. A *ring* $\langle R, +, \cdot\rangle$ is a set R together with two binary operations $+$ and $\cdot$, called *addition* and *multiplication*, respectively, defined on R such that:

$\mathcal{R}_1$**:** $\langle R, +\rangle$ is an abelian group.

$\mathcal{R}_2$**:** Multiplication $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

$\mathcal{R}_3$**:** For all $a, b, c \in R$, the *left distribution law* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the *right distribution law* $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Definition. If a and b are two nonzero elements of ring R such that $ab = 0$, then either a and b are *divisors of* 0. An *integral domain* D is a commutative ring with unity $1 \neq 0$ (that is, 1 is the multiplicative identity) and containing no divisors of 0.

Example. Consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, the integers modulo 6 (technically, the elements of $\mathbb{Z}_6$ are equivalence classes of integers...). Then $\langle \mathbb{Z}_6, +, \cdot \rangle$ is a ring. Elements $2, 3 \in \mathbb{Z}_6$ are divisors of 0 since $2 \cdot 3 = 0$.

Note. We claim that the space of analytic function on region G, $H(G)$, is an integral domain. Clearly, $H(G)$ is a ring. Also, for $f, g \in H(G)$, we do not have $fg \equiv 0$ (that is, fg is the function which is identically 0 on G) unless either $f \equiv 0$ or $g \equiv 0$. So $H(G)$ has no zero divisors and is an integral domain.

Definition. A ring is which multiplication is commutative is a *commutative ring*. An element of a ring with unity is a *unit* if the element has a multiplicative inverse in the ring. If every nonzero element of a ring is a unit, then the ring is a *division ring*. A *field* is a commutative division ring.

Note. We claim that the meromorphic functions on region G, $M(G)$, is a field. Clearly $M(G)$ is a ring (Corollary VII.5.20 is useful in proving closure). Similarly to the case for $H(G)$, $M(G)$ contains no zero divisors (and so $M(G)$ is also an integral domain). For any $f \in M(G)$, where $f \not\equiv 0$, we have that $1/f \in M(G)$ (the roots of f correspond to the poles of $1/f$ and the poles of f correspond to the roots of $1/f$). So $M(G)$ is a division ring. Of course multiplication is commutative, and so $H(G)$ is a field.

Note. Of course $H(G)$ is not a division ring, since $f(z) = 1/(z - a)$ where $a \in G$ has no multiplicative inverse in $H(G)$. We *can* say that $H(G)$ is a commutative integral domain, though.

Note. We need a theorem from modern algebra:

> **Theorem IV.21.5.** Any integral domain D can be enlarged to (or embedded in) a field F such that every element of f can be experssed as a quotient of two elements of D. Such a field F is a *field of quotients* of D.

See page 194 of Fraleigh's *A First Course in Abstract Algebra*, 7th edition (Addison-Wesley, 2003) and `http://faculty.etsu.edu/gardnerr/4127/notes/IV-21.pdf`.

Note. Corollary VII.5.20 shows that every $f \in M(G)$ satisfies $f = g/h$ for some $g, h \in H(G)$. Therefore, $M(G)$ is a field of quotients of integral domain $H(G)$.

Revised: 8/9/2017