VII.6. Factorization of the Sine Function

Note. In this brief section, we use the Weierstrass Factorization Theorem to write $\sin \pi z$ as an infinite product using the fact that its zeros are precisely the integers. Since we treat $z = 0$ differently from the other zeros of a function (because $E_0(z)$ is of a different form than the other $E_p(z)$), when summing or taking a product over the integers (the zeros of $\sin \pi z$), we pull $n = 0$ out and denote a sum or product over $\mathbb{Z} \setminus \{0\}$ with a prime: $\sum_{n=-\infty}^{\infty}{}' a_n - \sum_{n=1}^{\infty} a_{-n} + \sum_{n=1}^{\infty} a_n$ and $\prod_{n=-\infty}^{\infty}{}' a_n = \prod_{n=1}^{\infty} a_{-n} \prod_{n=1}^{\infty} a_n$ (we'll have the necessary absolute convergence to rearrange).

Theorem VII.6.A. For all $z \in \mathbb{C}$,

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

and the convergence is uniform over compact subset of $\mathbb{C}$.

Note. In Exercise VII.6.1, it is shown that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

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