VII.7. The Gamma Function

Note. In this section we define the meromorphic function $\Gamma(z)$ using the Weierstrass Factorization Theorem. We give some properties and an integral definition of the gamma function. In the next section, we find relationships between the gamma function and the famous zeta function.

Lemma VII.7.A. Let G be an open set in $\mathbb{C}$ and let $\{f_n\}$ be a sequence of analytic functions on G. Suppose $\{f_n\}$ converges to f (not identically 0) in $H(G)$. Then $\{f_n\}$ converges to f in $M(G)$ provided f is not the 0 function (the 0 function if not meromorphic since the zeros of a meromorphic function are isolated).

Note. Since $d(z_1, z_2) = d(1/z_1, 1/z_2)$ for all nonzero $z_1, z_2 \in \mathbb{C}$ and for $z \neq 0$, $d(z, 0) = d(1/z, \infty)$ (see Section VII.3) then $f_n \to f$ in $M(G)$ implies $1/f_n \to 1/f$ in $M(G)$. This observation, along with Lemma VII.7.A, allows us to take an analytic function which we write as a limit (an infinite product) using the Weierstrass Factorization Theorem and consider its reciprocal as an infinite product. In fact, with $1/f_n \to 1/f$ in $M(G)$, we have $1/f_n$ converges uniformly to $1/f$ on any compact set K on which no f_n vanishes.

Note. By Theorem VII.5.12, $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$ (here $\{z_n\} = \{-n\}_{n=1}^{\infty}$ and $p_n = 1$ for $n \in \mathbb{N}$) converges in $H(\mathbb{C})$ to an entire function which only has simple zeros at $z = -1, -2, \ldots$. So the reciprocal $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ converges on compact subsets of $\mathbb{C} \setminus \{-1, -2, \ldots\}$ to a function with simple poles at $z = -1, -2, \ldots$. We define this as the gamma function.

Definition VII.7.2. The *gamma function*, $\Gamma(z)$, is the meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$ defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

where γ is a constant such that $\Gamma(1) = 1$. The constant γ is the *Euler contant.*

Note. We need to establish that such a γ exists and find (or at least approximate) its value. Now as discussed above, $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$ is analytic and nonzero at $z = 1$, so $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ is nonzero at $z = 1$. So define $c = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}$. Since c is positive, let $\gamma = \log c$, Then γ will produce $\Gamma(1) = 1$. We have

$$c = e^{\log c} = e^{\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}.$$

Since all terms in the infinite product are positive real numbers, we can apply the usual natural logarithm on $\mathbb{R}$ to get

$$\begin{aligned}
\gamma &= \log \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n} = \sum_{k=1}^{\infty} \left(\left(1 + \frac{1}{k}\right)^{-1} e^{1/k} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right) \right) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log(k+1) + \log k \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \log(k+1) - \log k \right) \\
&= \lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \log(n+1) \right) \\
&= \lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) + (\log n - \log n) - \log(n+1) \right) \\
&= \lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \log \frac{n+1}{n} - \log n \right)
\end{aligned}$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right) \text{ since } \lim_{n\to\infty}\log\frac{n+1}{n}=0.$$

So we can express γ as

$$\gamma = \lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right).$$

In Exercise VII.7.1, it is to be shown that $\gamma \in (0,1)$. It is commented that: "It is unknown whether γ is rational or irrational" (page 185).

Theorem VII.7.B. Gauss's Formula.

For $z \neq 0, -1, -2, \ldots$ we have

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

Theorem VII.7.C. Functional Equation.

For $z \neq 0, -1, -2, \ldots$, $\Gamma(z+1) = \Gamma(z)$.

Note. For $n \in \mathbb{N}$, we get from Theorem VII.7.C that $\Gamma(n+1) = n!$, so we see that the gamma function is a generalization of the factorial function on $\mathbb{N}$. In fact, for $z \in \mathbb{C}$, $z \neq -1, -2, -3, \ldots$ we could define $z! = \Gamma(z+1)$. You may recall encountering the gamma function defined for $z > 0$ as an integral in Calculus 2. More of this soon...

Lemma VII.7.D. The residue of the gamma function Γ at simple pole $-n$, $n \in \mathbb{N}\cup\{0\}$, is $\mathrm{Res}(\Gamma; -n) = (-1)^n/n!$.

Lemma VII.7.E. $\log \Gamma(x)$ is a convex function for $x > 0$.

Note. It turns out that the Functional Equation (Theorem VII.7.C) for $x > 0$, Lemma VII.7.E, the value of Γ at 1, and the nonnegativity of Γ for $x > 0$, completely classifies the gamma function for $x > 0$, as shown in the following theorem. It is named for Danish mathematicians Harald Bohr (1887–1951; brother of physicist Niels Bohr and a member of the Danish national football team for the 1908 Summer Olympics, where he won a silver medal) and Johannes Mollerup (1872–1937). Their result appeared in *Textbook of Mathematical Analysis*, which was published around 1915.

Theorem VII.7.13. Bohr-Mollerup Theorem.
Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties:

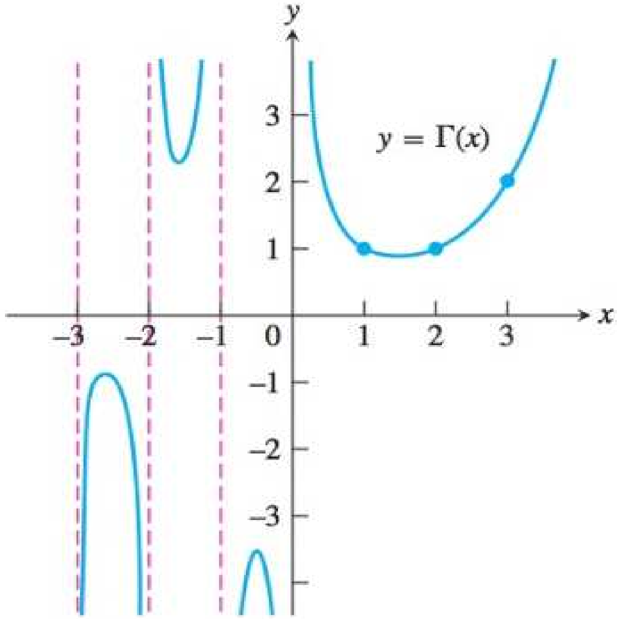
(a) $\log f(x)$ is a convex function;

(b) $f(x+1) = xf(x)$ for all $x > 0$;

(c) $f(1) = 1$.

Then $f(x) = \Gamma(x)$ for $x > 0$.

Note. You may have encountered the gamma function in Calculus 2 when dealing with integration by parts. On pages 510 and 511 of *Thomas' Calculus Early Transcendentals*, 12th edition (G. Thomas, M. Wier, J. Hass, Addison-Wesley, 2010) the gamma function is defined as the real function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ for $x > 0$ and the reader is asked to prove that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N} \cup \{0\}$. Thomas gives the following graph (Figure 8.21, page 511):



In addition, on page 918 it is outlined how to prove that $\Gamma(1/2) = \sqrt{\pi}$. The following is a complex version of the Calculus 2 approach.

Theorem VII.7.15. If $\text{Re}(z) > 0$ then $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$.

Note. To prove Theorem VII.7.15, we need three results.

Lemma VII.7.16. Let $S = \{z \mid a \le \text{Re}(z) \le A\}$ where $0 < a < A < \infty$.

(a) For every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in S$, $\left|\int_\alpha^\beta e^{-t}t^{z-1}\,dt\right| < \varepsilon$ whenever $0 < \alpha < \beta < \delta$.

(b) For every $\varepsilon > 0$ there is a number κ such that for all $z \in S$, $\left|\int_\alpha^\beta e^{-t}t^{z-1}\,dt\right| < \varepsilon$ whenever $\beta > \alpha > \kappa$.

Note. Conway state that parts (a) and (b) of Lemma VII.7.16 "embody exactly the concept of a uniformly convergent integral. (see page 181). In the next result, we use this idea to establish that a sequence is Cauchy in $H(G)$ (and hence convergent since $H(G)$ is complete by Corollary VII.2.3).

Proposition VII.7.17. If $G = \{z \mid \text{Re}(z) < 0\}$ and

$$f_n(z) = \int_{1/n}^{n} e^{-t}t^{z-1}\,dt$$

for $n \in \mathbb{N}$ and $z \in G$, then each f_n is analytic on G and the sequence is convergent on $H(G)$.

Note. Theorem VII.7.15 claims that the limit of the sequence $\{f_n\}$ of Proposition VII.7.17 is the gamma function $\Gamma(z)$. We need one more lemma before proving this.

Lemma VII.7.19.

(a) The sequence $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$.

(b) If $t \geq 0$ then $(1-t/n)^n \leq e^{-1}$ for all $n \geq t$.

Note. We now have the equipment for the proof of Theorem VII.7.15.

Note. By Exercise VII.7.2, $\Gamma(1/2)=\sqrt{\pi}$. So by Theorem VII.7.15 we have

$$\sqrt{\pi}=\int_0^\infty e^{-t}t^{-1/2}\,dt.$$

With a change of variables of $t=s^2$ gives

$$\sqrt{\pi}=\int_0^\infty e^{-s^2}s^{-1}[2s]\,ds=2\int_0^\infty e^{-s^2}\,ds,$$

or

$$\int_0^\infty d^{-s^2}\,ds=\frac{\sqrt{\pi}}{2}.$$

The standard way to establish this integral in real variables, is to convert from rectangular to polar coordinates (see Exercise 41 in Section 15.4 on pages 876 and 877 of *Thomas' Calculus*, 12th edition). This is an important integral since it allows one to prove that the total area under the normal distribution is 1:

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx=1.$$