VIII.2. Simple Connectedness.

Note. In this section we state a single result which ties together several approaches to simple connectedness. The one result unites topological, algebraic, and analytic ideas of simple connectedness. This section is largely independent of Section VIII.1, but does reference Proposition VIII.1.1 and Corollary VIII.1.15 (which can be read independently of covering Section VIII.1).

Definition VIII.2.1. Let X and Ω be metric spaces. A *homeomorphism* between X and Ω is a continuous map $f : X \to \Omega$ which is one to tone, onto, and such that $f^{-1} : \Omega \to X$ is continuous. If there is a homeomorphism between X and Ω then the metric spaces X and Ω are *homeomorphic*.

Note. A homeomorphism is sort of like an isomorphism between metric spaces. It does not necessarily preserve distances (as an "isometry" does) by it does preserve open sets which means that it preserves all things topological such as limits, continuity, compactness, etc. Notice that for homeomorphism f, f maps open sets to open sets (since f^{-1} is continuous) and f^{-1} of an open set is open (since f is continuous). That is, f and f^{-1} are "open maps" (see page 99).

Example. $\mathbb{C}$ and $D = \{z \mid |z| < 1\}$ are homeomorphic. This is because $f : \mathbb{C} \to D$ defined as $f(z) = \dfrac{z}{1 + |z|}$ is a homeomorphism with $f^{-1} : D \to \mathbb{C}$ as $f^{-1}(\omega) = \frac{\omega}{1-|\omega|}$.

Theorem VIII.2.2. Let G be an open connected subset of $\mathbb{C}$. The the following are equivalent:

(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} \setminus G$;

(c) $\mathbb{C}_\infty \setminus G$ is connected;

(d) For any $f \in H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;

(e) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;

(f) Every function $f \in H(G)$ has a primitive;

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

(i) G is homeomorphic to the unit disk;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Note. Conway gets a little emotional on page 204 and states: "This theorem constitutes an aesthetic peak in Mathematics. Notice that it says that a topological condition (simple connectedness) as equivalent to analytic conditions (eg., the existence of harmonic conjugates and Cauchy's Theorem) as well as an algebraic condition (the existence of a square root) and other topological conditions." The topological connection is a deep one. In James Munkres' *Topology*, 2nd edition (Prentice Hall, 2000), Sections 65 and 66 are titled "The Winding Number of a Simple Closed Curve" and "The Cauchy Integral Formula," respectively. These sections appear in Chapter 10, "Separation Theorems in the Plane."

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