VIII.3. Mittag-Leffler's Theorem.

Note. In the Weierstrass Factorization Theorem (Theorem VII.5.14) an analytic function with given zeros is written as a limit of a product of polynomials (with an exponential function thrown into the product). In this section we state a related result where we are given a set of points and we create a meromorphic function with each of the points as a pole and the singular part at each pole is of any desired (admissible) form. The existence of such a function is guaranteed by the Mittag-Leffler Theorem.

Note. Let G be an open subset of $\mathbb{C}$ and let $\{a_k\}$ be a sequence of distinct points in G such that $\{a_k\}$ has no limit point in G. For each $k \geq 1$ consider the rational function

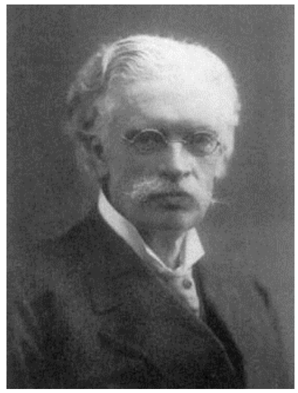
$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j} \tag{3.1}$$

where $m_k \in \mathbb{N}$ and $A_{1k}, A_{2k}, \ldots, A_{m_k k}$ are any complex numbers. We desire to find a meromorphic function f on G with poles exactly at the points $\{a_k\}$ and such that the singular part of f at a_k is $S_k(z)$.

Theorem VIII.3.2. Mittag-Leffler's Theorem.

Let G be an open set, $\{a_k\}$ a sequence of distinct points in G without a limit point in G, and let $\{S_k(z)\}$ be the sequence of rational functions given by equation (3.1). Then there is a meromorphic function f on G whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at a_k is $S_k(z)$.

Note. Theorem VIII.3.2 is named for Swedish mathematician Gösta Mittag-Leffler.



March 16, 1846–July 7, 1927

Mittag-Leffler spent most of his professional career in Sweden, but did travel abroad. In 1875 we went to Berlin and attended lectures by Weierstrass which were to be influential in his future work. He became chair of the newly opened university in Stockholm in 1881. He organized the creation of the journal *Acta Mathematica*. The Mittag-Leffler Theorem (Theorem IX.3.2) is like Weierstrass's Theorem (Theorem VII.5.14) for entire functions with given zeros, but for meromorphic functions with given poles. He published this work in 1884 (in French) as "Sur la représentation analytique des fonctions monogènes uniformes dune variable indépendante," *Acta Mathematica*, **4** (1884), 1–79; a copy is available online at Project Euclid: `https://projecteuclid.org/download/pdf_1/euclid.acta/1485803325`. Mittag-Leffler was one of the first mathematicians to support Cantor's then-controversial theory of sets. The photo and these notes are based on the biography of Mittag-Leffler on the MacTutor History of Mathematics archive at: `www-history.mcs.st-andrews.ac.uk/Biographies/Mittag-Leffler.html`.

Example. We now find a meromorphic function with a pole at each $n \in \mathbb{Z}$. We take the singular part to be $(z-n)^{-1}$. But $\sum_{n=-\infty}^{\infty} 1/(z-n)$ does not converge in $M(\mathbb{C})$. But $\dfrac{1}{z-n} + \dfrac{1}{z+n} = \dfrac{2z}{z^2-n^2}$, and $\dfrac{1}{z} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2z}{z^2-n^2}$ does converge in $M(\mathbb{C})$. By Exercise V.2.8 we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

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