Chapter XI. Entire Functions

Note. In this chapter, we give a few results concerning functions which are analytic in the entire complex plane. In the first section, we give a result with applications to the study of polynomials (this application is not mentioned in Conway). In the second section we give parameters related to the rate of growth of entire functions. These results are, in a sense, related to the idea of the degree of a polynomial. In the third section, the parameters of section XI.2 are further explored and the range of an entire function is addressed. A more detailed study of entire functions is given in the classical text *Entire Functions* by Ralph Boas, Academic Press Pure & Applied Mathematics, Number 5 (1954).

XI.1. Jensen's Formula.

Note. The Mean Value Theorem (Theorem X.1.4) states:

> If $u : G \to \mathbb{R}$ is a harmonic function and $\overline{B}(a;r)$ is a closed disk contained in G, then
>
> $$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta.$$

If f is analytic in an open set containing $\overline{B}(0;r)$ and f doesn't vanish in $\overline{B}(0;r)$ then $\log|f|$ is harmonic on $\overline{B}(0;r)$. So by the Mean Value Theorem,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta. \qquad (1.1)$$

Note. In fact, (1.1) holds even if f has zeros on $|z| = r$, as we show next. The main result of this section, Jensen's Formula, deals with an extension of (1.1) to functions f which have zeros in $B(0; r)$.

Lemma XI.1.A. If f is analytic in an open set containing $\overline{B}(0; r)$ and f doesn't vanish in $B(0; r)$ then (1.1) holds.

Note. We now give a result related to Lemma XI.1.A, but one which allows f to have zeros in $B(0; r)$. As seen in the proof of Lemma XI.1.A, we need only consider a finite number of such zeros by Theorem IV.3.7.

Theorem XI.1.2. Jensen's Formula.
Let f be an analytic function on a region containing $\overline{B}(0; r)$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $B(0; r)$ repeated according to multiplicity. If $f(0) \neq 0$ then

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Note. Jensen's Formula is named for Danish mathematician Johan Ludwig Jensen (May 8, 1859–March 5, 1925). Jensen was essentially self taught in research level mathematics and never held an academic appointment; so he was an amateur mathematician! Most of his career, he worked for the Copenhagen Telephone Company. He proved his formula while studying the Riemann hypothesis (which is stated in

Section VII.8). His work appeared in "Sur un nouvel et important théorème de la théorie des fonctions," *Acta Mathematica*, **22**(1), (1899) 359-364.

May 8, 1859–March 5, 1925

Jensen published a result on convex functions in 1906 (also in *Acta Mathematica*). We see this result in the graduate Real Analysis sequence (MATH 5210/5220); see my online notes on "Convex Functions" at `http://faculty.etsu.edu/gardnerr/5210/notes/6-6.pdf`. The photo and these notes are based on the biography of Jensen's on the MacTutor History of Mathematics archive at: `http://www-history.mcs.st-andrews.ac.uk/Biographies/Jensen.html`.

Note. An application of Jensen's Formula to counting the number of zeros of a polynomial in a disk is given by Edward Titchmarsh in his *Theory of Functions*, 2nd Edition, Oxford University Press (1939); see page 280. This book is online at `https://archive.org/details/TheTheoryOfFunctions`.

Theorem XI.1.B. Titchmarsh's Number of Zeros Theorem.
Let f be analytic in $|z| < R$. Let $|f(z)| \leq M$ in the disk $|z| \leq R$ and suppose $f(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

Note. I have used Titchmarsh's Number of Zeros Theorem in some of my research on polynomials. An example of this work is the following:

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where for some $t > 0$ and some $0 \leq k \leq n$,

$0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{k-1} \leq |a_k| \geq |a_{k+1}| \geq \cdots \geq |a_{n-1}| \geq |a_n|$

and $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for $1 \leq j \leq n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P in the disk $|z| \leq \delta$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = |a_0|(1 - \cos\alpha - \sin\alpha) + 2|a_k|\cos\alpha + |a_n|(1 + \sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|$.

This is Corollary 1 of "The Number of Zeros of a Polynomial in a Disk" by Robert Gardner and Brett Shields, *Journal of Classical Analysis*, **3**(2) (2013), 167–176. A copy is online at `http://faculty.etsu.edu/gardnerr/pubs/JCA-03-15.pdf`. Brett Shields was one of my graduate students here in the ETSU master's math program.

Note. If instead of using the Mean Value Theorem (Theorem X.1.4), we use Corollary X.2.9 and apply it to harmonic function $\log|f|$, we can produce analogous results to (1.1), Lemma XI.1.A, and Jensen's Formula, a proof of which is to be given in Exercise XI.1.A.

Theorem XI.1.3. The Poisson-Jensen Formula.
Let f be analytic in a region which contains $\overline{B}(0;r)$ and let $a_1, a_2, \ldots, a_n$ be the zeros of f in $B(0;r)$ repeated according to multiplicity. If $|z| < r$ and $f(z) \neq 0$ then

$$\log|f(z)| = -\sum_{k=1}^{n} \log\left|\frac{r^2 - \overline{a_k}z}{r(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|f(re^{i\theta})|\,d\theta.$$

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