# XI.3. Hadamard's Factorization Theorem.

**Note.** In this section we give a converse of Corollary XI.2.6. That is, we prove that a function of finite order is of finite genus. We know that a function of finite genus can be factored as $z^n e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)$ where $g$ is a polynomial of degree at most $\mu$. This is why the result is called a "factorization theorem."

**Lemma XI.3.1.** Let $f$ be a nonconstant entire function of order $\lambda$ with $f(0) \neq 0$ and let $\{a_1, a_2, \ldots\}$ be the zeros of $f$ repeated according to multiplicity and arranged so that $|a_1| \le |a_2| \le \cdots$. If $p$ is an integer such that $p > \lambda - 1$ then

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$$

for $z \in \{a_1, a_2, \ldots\}$.

**Note.** The proof of Lemma XI.3.1 assumes that $f$ has infintely many zeros, but also holds if $f$ only has finitely many zeros.

**Theorem XI.3.4. Hadamard's Factorization Theorem.**
If $f$ is an entire function of finite order $\lambda$ then $f$ has finite genus $\mu \le \lambda$. Therefore, $f$ can be factored as $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)$ where $g$ is a polynomial of degree at most $\mu$.

**Note.** Picard's Theorems, to be seen in Sections XII.2 and XII.4, concern the range of analytic functions. We can use Hadamard's Factorization Theorem to prove a special case of Picard's Theorem.

**Theorem XI.3.6.** If $f$ is an entire function of finite order, then $f$ assumes each complex number with one possible exception.

**Note.** The exponential function $f(z) = e^z$ assumes every value except 0 (see Lemma III.2.A(b)), since a branch of the logarithm can be defined on some region containing any given nonzero complex number. In fact, since $e^z$ is periodic with period $2\pi i$, it assumes each nonzero value an infinite number of times. The following theorem and corollary shows that a similar result holds for certain entire functions of finite order.

**Theorem XI.3.7.** Let $f$ be an entire function of finite order $\lambda$ where $\lambda$ is not an integer. Then $f$ has infinitely many zeros.

**Note.** If $\alpha \in \mathbb{C}$, then applying Theorem XI.3.7 to $f(z) - \alpha$ we see that $f$ assumes the value $\alpha$ an infinite number of times. This is summarized in the following.

**Corollary XI.3.8.** If $f$ is an entire function of order $\lambda$ and $\lambda$ is not an integer then $f$ assumes each complex value an infinite number of times.

*Revised: 11/4/2017*