XII.2. The Little Picard Theorem.

Note. Recall that Theorem XI.3.6 states that an entire function of finite order assumes every complex number with one possible exception. In this section we weaken the hypotheses of Theorem XI.3.6 significantly to show in the Little Picard Theorem that every nonconstant entire function (regardless of order) assumes every complex number with one possible exception. Of course, $f(z) = e^z$ is an example of such a function with a range omitting one complex number. The proofs in this section require one result from the previous section, namely Corollary XII.1.11. We also need two lemmas which we now present.

Lemma XII.2.1. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Then there is an analytic function g on G such that $f(z) = -\exp(i\pi\cosh(2g(z))$ for all $z \in G$.

Lemma XII.2.2. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Let g be analytic on G where $f(z) = -\exp(i\ pi \cosh(2g(z)))$ for all $z \in G$ (such g exists by Lemma XII.2.2). Then $g(G)$ contains no disk of radius 1.

Note. With Lemmas XII.2.1 and XII.2.2, along with Corollary XII.1.11, we are equipped to prove the Little Picard Theorem.

Theorem XII.2.3. The Little Picard Theorem.

If f is an entire function that omits two values then f is constant. That is, if f is a nonconstant entire function then it assumes every complex number with one possible exception.

Note. The Great Picard Theorem (Theorem XII.4.2) addresses the behavior of an analytic function in the deleted neighborhood of an essential singularity. It claims that the function assumes every complex number, with one possible exception, an infinite number of times.

Note. The Little Picard Theorem allows us to give yet another proof of the Fundamental Theorem of Algebra.

Theorem XII.2.A. Fundamental Theorem of Algebra.

If p is a nonconstant polynomial then there is a complex number a with $p(a) = 0$.

Proof. Let $p(z) = \sum_{n=0}^{n} a_n z^n$. Then

$$\begin{aligned} \lim_{z\to\infty} |p(z)| &= \lim_{z\to\infty} |z^n(a_n + a_{n-1}z + \cdots |z_0 z^{-n})| \\ &= \lim_{z\to\infty} |z|^n \lim_{z\to\infty} |z_n + z_{n-1}z^{-1} + \cdots + a_0 z^{-n}| = \infty. \end{aligned}$$

So there exists $R > 0$ such that for all z with $|z| > R$ we have $|p(z)| > 1$.

ASSUME $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then by the Little Picard Theorem (Theorem XII.2.3), p must assume every nonzero value in $\mathbb{C}$. Consider the set $S = \{1/k \mid k \in \mathbb{N}\}$. Then f must attain each value in set S and $S \subset p(\overline{B}(0, R))$. Since f is continuous and $\overline{B}(0, R)$ is compact by the Heine-Borel Theorem (Theorem II.4.10), then $p(\overline{B}(0, R))$ is compact by Theorem II.5.8(a). So $p(\overline{B}(0, R))$ is closed by Proposition II.4.3 and hence includes its limit points. But $S \subset p(\overline{B}(0,R))$ and 0 is a limit point of S, so 0 is a limit point of $p(\overline{B}(0,R))$. So $0 \in p(\overline{B}(0,R))$ and $p(a) = 0$ for some $a \in \overline{B}(0,R) \subset \mathbb{C}$, a CONTRADICTION to the assumption that p is never 0. So the assumption is false and the claim follows. ∎

Note. The Little Picard Theorem (and the Great Picard Theorem) is named from Charles Émile Picard (July 24, 1856 to December 11, 1941). At age 23 in 1879 he published the Little Picard Theorem in *Sur une propriété des fonctions entières*, *Comptes rendus de l'Académie des Sciences, Paris* **88** (1879), 1024–1027.

Picard made his most important contributions in the fields of analysis, function theory, differential equations, and analytic geometry. He used methods of iteration to show the existence of solutions of ordinary differential equations in what is today known as the method of Picard iterates. He made contributions to algebraic geometry, building on the work of Abel and Riemann. He also applies his work to problems in elasticity, heat, and electricity. (This note is based on the "MacTutor History of Mathematics archive" biography of Émile Picard at `http://www-history.mcs.st-and.ac.uk/Biographies/Picard_Emile.html`.)

Revised: 12/15/2017