# XII.3. Schottky's Theorem.

**Note.** Schottky's Theorem shows the existence of a bound on a function which does not take on the values 0 or 1. The bound is valid on $|z| \le \beta$ for a given $0 \le \beta < 1$ and so gives a bound on proper subdisks of the unit disk. Schottky's Theorem is used in the proof of the Great Picard Theorem (namely, in the proof of Theorem XII.4.1, the Montel-Carathédory Theorem).

**Note.** In Lemma XII.2.1, we considered a simply connected region on which an analytic function $f$ does not assume the values 0 or 1. In the proof, we used a branch of $\log(f)$, denoted $\ell(z)$, we set $F(z) = \ell(z)/(2\pi i)$, we set $H(z) = \sqrt{F(z)} - \sqrt{F(z) - 1}$, and let $g$ be a branch of $\log(H(z)$. In Schottky's Theorem, we use specific branches of $\log(g)$ and $\log(H)$. Specifically, we require $0 \le \mathrm{Im}(\ell(0)) < 2\pi$ and $0 \le \mathrm{Im}((g(0)) < 2\pi$.

**Theorem XII.3.1. Schottky's Theorem.**
For each $\alpha$ and $\beta$ with $0 < \alpha < \infty$ and $0 \le \beta < 1$, there is a constant $C(\alpha, \beta)$ such that $f$ is an analytic function on some simply connected region containing $\overline{B}(0; 1)$ that omits the values 0 and 1, and such that $|f(0)| \le \alpha$, where $|f(z)| \le C(\alpha, \beta)$ for $|z| \le \beta$.

**Note.** Applying Schottky's Theorem to $f(Rz)$ we get the following.

**Corollary XII.3.7.** Let $f$ be analytic on a simply connected region containing $\overline{B}(0; R)$ and suppose that $f$ omits the values 0 and 1. If $C(\alpha, \beta)$ is the constant obtained in Schotkky's Theorem and $|f(z)| \le \alpha$ then $|f(z)| \le C(\alpha, \beta)$ for $|z| \le \beta R$.

**Note.** In the proof of the Great Picard Theorem in the next section, we'll see how Schottky's Theorem is used to show that a certain family of functions is uniformly bounded and so (by Montel's Theorem, Theorem VII.2.9) is normal.

**Note.** Schottky's Theorem is named for Friedrich Schottky (July 24, 1851 to August 12, 1935). He published it in "Über den Picardschen Satz und die Borelschen Ungleichungen," *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, (1904), 1244-1263.

Schottky obtained his doctorate in 1875 under the direction of Karl Weierstrass and others. He was with the University of Breslau from 1878 to 1882, then he moved to Zurich, Switzerland. He returned to the University of Berlin where he was chair and where he retired in 1922. His doctoral thesis was on conformal mapping, but most of his work concerned elliptic, abelian, and theta functions. His son, Walter Schottky, was a doctoral student of Max Planck and made important contributions to the theory of the electron. (This note is based on the "MacTutor History of Mathematics archive" biography of Friedrich Schottky at `http://www-history.mcs.st-andrews.ac.uk/Biographies/Schottky.html`.)

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