XII.4. The Great Picard Theorem.

Note. We conclude the text by applying some of the theory developed in Chapter VII and this chapter. We use results from Sections VII.1 (namely, Montel's Theorem and a corollary to Hurwitz's Theorem) and Section XII.3 (Corollary 7 and Schottsky's Theorem). We need a preliminary result.

Theorem XII.4.1. Montel-Carathéodory Theorem.
If $\mathcal{F}$ is the family of all analytic functions on a region G that do not assume the values 0 and 1, then $\mathcal{F}$ is normal in $C(G, \mathbb{C}_\infty)$.

Theorem XII.4.2. The Great Picard Theorem.
Suppose an analytic function f has an essential singularity at $z = a$. Then in each neighborhood of a, f assumes each complex number, with one possible exception, an infinite number of times.

Note. We can also state the Great Picard Theorem as follows.

Corollary XII.4.3. If f has an isolated singularity at $z = a$ and if there are two complex numbers that are not assumed infinitely often by f then $z = a$ is either a pole or a removable singularity.

Note. In conclusion, we have a final result concerning entire functions.

Corollary XII.4.4. If f is an entire function that is not a polynomial then f assumes every complex number, with one possible exception, an infinite number of times.

Note. Corollary XII.4.4 is a generalization of the Little Picard Theorem, since Corollary XII.4.4 addresses how often every complex number is assumed (with one possible exception).

Note. Now that we've completed Conway's book, it is with some hesitation I point out that this was just the first volume of a two volume series! The reference for the 400 page second volume is: John B. Conway *Functions of One Complex Variable II*, Springer-Verlag (1995). The table of contents is:

The journey continues. . .