Section 9.7. Periodic Solutions and Limit Cycles

Note. In this section we consider the autonomous system $\vec{x}' = \vec{f}(\vec{x})$ and look for periodic solutions of the form $\vec{x}(t + T) = \vec{x}(t)$. These solutions will be closed curves in the phase plane. We saw examples of this in the linear system $\vec{x}' = A\vec{x}$ where the eigenvalues of $A$ were purely imaginary. We also saw this in the nonlinear predator-prey equations.

Note. The following example illustrates a system with periodic solutions which also shows a certain stability.

Example. Page 492 Example 1. Consider the system
\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix}
y + x - x(x^2 + y^2) \\
-x + y - y(x^2 + y^2)
\end{bmatrix}.
\]
Discuss the solutions.

Solution. Well, $(0, 0)$ is certainly a critical point. The associated almost linear system is
\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
where $A$ has eigenvalues $1 \pm i$. So $(0, 0)$ is an unstable spiral point of the almost spiral linear system and therefore of the original system. Now from the original system, we have
\[
x \frac{dx}{dt} = xy + x^2 - x^2(x^2 + y^2)
\]
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\[ y \frac{dy}{dt} = -xy + y^2 - y^2(x^2 + y^2) \]

and so

\[ x \frac{dx}{dt} + y \frac{dy}{dt} = (x^2 + y^2) - (x^2 + y^2)^2. \]

Letting \( x = r \cos \theta \) and \( y = r \sin \theta \) we have \( r^2 = x^2 + y^2 \) and \( r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \) and

\[ r \frac{dr}{dt} = r^2 - r^4 = r^2(1 - r^2). \]

So, \( dr/dt = 0 \) when \( r = 0 \) and when \( r = 1 \) (here, we keep \( r \geq 0 \) in \((r, \theta)\) polar coordinates). Also, \( dr/dt > 0 \) if \( r < 1 \) and \( dr/dt < 0 \) if \( r > 1 \). Now for \( \theta \) consider

\[ y \frac{dx}{dt} - x \frac{dy}{dt} = r \sin \theta \left( \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt} \right) - r \cos \theta \left( \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt} \right) = -r^2 \frac{d\theta}{dt}. \]

Also, as above,

\[ y \frac{dx}{dt} - x \frac{dy}{dt} = x^2 + y^2. \]

So \(-r^2 d\theta/dt = r^2\) and \(d\theta/dt = -1\). Solving (*) we get

\[ r \frac{dr}{dt} = r^1(1 - r^2) \text{ or } \frac{dr}{r(1 - r^2)} = dt \]

which gives

\[ r = \frac{1}{\sqrt{1 + (1/\rho^2 - 1)e^{-2t}}} \text{ and } \theta = -t + \alpha \]

where \( r(0) = \rho \) and \( \theta(0) = \alpha \). Notice that if \( \rho < 1 \) then \( r \to 1 \) as \( t \to \infty \), and if \( \rho > 1 \) then \( r \to 1 \) as \( t \to \infty \). The trajectories are (this is Figure 9.7.1 in the 10 edition of DiPrima and Boyce):
**Definition.** A closed trajectory in the phase plane such that other trajectories spiral toward it (either from the inside or outside) as \( t \to \infty \) is called a *limit cycle*.

**Definition.** If all the trajectories near a limit cycle (both those inside and outside) spiral towards the limit cycle as \( t \to \infty \), then the limit cycle is said to be *stable*. If the trajectories on one side spiral towards and on the other side spiral away, then the limit cycle is *semistable*. If the trajectories on both sides of a closed trajectory spiral away as \( t \to \infty \), then the closed trajectory is *unstable* (it’s not even called a limit cycle). In the case that nearby trajectories neither approach nor depart a closed trajectory, it is called *neutrally stable*. 
Note. Now rewrite $\vec{x}' = \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix}$.

Theorem 9.7.1. Let the functions $F$ and $G$ have continuous first partial derivatives in a domain $D$ of the $xy$-plane. A closed trajectory must necessarily enclose at least one critical point. If it encloses only one critical point, then the critical point cannot be a saddle point.

Theorem. Poincaré Bendixson.

Let $F$ and $G$ have continuous first partial derivatives in a domain $D$ of the $xy$-plane. Let $D_1$ be a bounded subdomain in $D$ and let $R$ be the region that consists of $D_1$ and its boundary. Suppose that $R$ contains no critical point. If $x = \varphi(t), y = \psi(t)$ is a solution for all $t \geq t_0$, then either:

1. $x = \varphi(t), t = \psi(t)$ is a periodic solution, or

2. $x = \varphi(t), y = \psi(t)$ spirals towards a closed trajectory as $t \to \infty$.

In either case, $R$ contains a periodic solution.

Example. Page 492 Example 1. Consider (again) the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} y + x - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{bmatrix}.$$  

Apply the Poincaré-Bendixson Theorem to show that this system has a periodic solution.
**Solution.** We saw above that $dr/dt = r(1 - r^2)$. Now, in the region $\{r, \theta\} | 1/2 < r < 3/2\}$, $\left.\frac{dr}{dt}\right|_{r=1/2} > 0$ and $\left.\frac{dr}{dt}\right|_{r=3/2} < 0$ so any trajectory in the region remains in the region. Hence, by the Poincaré-Bendixson Theorem, the region must contain a periodic solution.

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