Chapter 1. Differential Equations and Their Solutions

Section 1.1. Classification of Differential Equations: Their Origin and Application

Note. In this section we give some preliminary definitions and examples.

Definition. An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a *differential equation* or "DE."

Definition. A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an *ordinary differential equation*, or "ODE."

Note. Examples of ODEs are:

Bessel's equation: $x^2 f''(x) + xf'(x) + (x^2 - p^2)f(x) = 0$.

Logistic equation: $f'(x) = x(k - x)$.

Definition. A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a *partial differential equation* or "PDE."

Note. Some examples of PDEs are:

The heat equation (without generation; i.e., without degradation of the medium) in one dimension:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \text{ where } u(x,t) \text{ is temperature and } k \text{ is diffusivity.}$$

The heat equation in 3 dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}.$$

The wave equation:

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} + \rho g \text{ where } T \text{ is a contant, } g \text{ is the gravitational constant,}$$

$$\rho \text{ is linear distance, and } u(x,t) \text{ is displacement.}$$

Definition. The order of the highest order derivative involved in a differential equation is called the *order* of the differential equation.

Note. The heat and wave equations are of order 2.

Definition. A *linear ordinary differential equation* of order n is an equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y + a_n(x)y = b(x)$$

where $a_0 \neq 0$ and $y = f(x)$.

Definition. A *nonlinear ordinary differential equation* is an ODE that is not linear.

Note. We will deal primarily with linear ODEs. Nonlinear differential equations are the target of a great deal of research to this day (in the areas of dynamical systems, fractals, etc.). In general, nonlinear DEs cannot be *precisely* solved.

Note. A (famous) nonlinear equation is the Lotka-Volterra predator/prey system of equations. Let x be the number of prey and y the number of predators. Then

$$\frac{1}{x}\frac{dx}{dt} = a - by \text{ and } \frac{1}{y}\frac{dy}{dt} = -c + dx,$$

$$\text{or } \dot{x} = x(a - by) \text{ and } \dot{y} = y(-c + dx).$$

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