Section 1.3. Initial-Value Problems, Boundary-Value Problems and Existence of Solutions

Note. In this section we will consider a DE with extra supplementary condition(s).

Definition. If the supplementary conditions of a DE relate to *one* x value, the problem is called an *initial-value problem* (IVP).

Example. Solve the IVP $\dfrac{d^2y}{dx^2} + y = 0$ and $\begin{cases} y(0) = 1 \\ y'(0) = 2. \end{cases}$

HINT: $y = c_1 \cos x + c_2 \sin x$ satisfies the DE $y'' + y = 0$.

Definition. If the supplementary conditions of a DE relate to two different x values, the problem is called a *boundary-value problem*, or BVP.

Example. Solve the BVP $\dfrac{d^2y}{dx^2} + y = 0$ where $y(0) = 1$ and $y(\pi/2) = 1$.

Note. We now consider first order IVPs in more detail.

Definition. Consider the first order DE: $y' = f(x, y)$ where f is a continuous function of x and y "near" the point (x_0, y_0). The *initial value problem* associated with this DE is to find a function $\varphi(x)$ satisfying the DE, defined "near" x_0 and satisfying the *initial condition* $\varphi(x_0) = y_0$.

Note. We are interested in whether an IVP has any solutions or maybe multiple solutions.

Theorem 1.1. Basic Existence and Uniqueness Theorem.
Consider the DE $\dfrac{dy}{dx} = f(x, y)$ where f is continuous on some open connected set D in the xy-plane and $\partial f/\partial y$ is also continuous on D. Let $(x_0, y_0) \in D$. Then there is a unique solutions φ of the DE defined near x_0 that satisfies the initial condition $\varphi(x_0) = y_0$.

Example. Consider the IVP $\dfrac{dy}{dx} = y^{1/3}$, $y(0) = 0$. Notice $f(x, y) = y^{1/3}$ and $\dfrac{\partial f}{\partial y} = \dfrac{1}{3y^{2/3}}$. So $\partial f/\partial y$ does not exist for $y = 0$. So a unique solution is not guaranteed. In fact, this IVP has infinitely many solutions (see page 19).

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