Chapter 2. First Order Equations for which Exact Solutions are Obtainable

Note. In this chapter, we cover some methods to solve first order equations.

Section 2.1. Initial-Value Problems, Boundary-Value Problems and Existence of Solutions

Note. In this section we define and solve exact DEs.

Note. Notice that the DE mentioned in the "Basic Existence and Uniqueness Theorem" $\dfrac{dy}{dx} = f(x,y)$ can be written in the differential form $M(x,y)\,dx + N(x,y)\,dy = 0$. Of course this process is reversible.

Definition. Let F be a function of two real variables such that F has continuous first partial derivatives in an open connected set (or domain) D. The *total differential* dF of the function F is defined as

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy$$

for all $(x,y) \in D$.

Example. Find the total differential of $F(x,y) = y^2 \cos x$.

Definition. The expression $M(x, y)\,dx + N(x, y)\,dy$ is called an *exact differential* is a domain D if there is a function $F(x, y)$ such that $dF(x, y)$ equals the expression for all $(x, y) \in D$. If the expression is an exact differential, then the DE $M(x, y)\,dx + N(x, y)\,dy = 0$ is called an *exact differential equation*.

Note. The function $F(x, y)$ satisfies the conditions

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \text{ and } \frac{\partial F(x, y)}{\partial y} = N(x, y).$$

This observation gives an easy way to check if a DE is exact.

Theorem 2.1. Consider the DE $M(x, y)\,dx + N(x, y)\,dy = 0$ where M and N have continuous first partial derivatives at all points in a rectangular domain D. The DE is exact if and only if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \text{ for all } (x, y) \in D.$$

Note. We see in the proof of Theorem 2.1 in the book that

$$F(x, y) = \int M(x, y)\,\partial x + \int \left[N(x, y) - \int \frac{\partial M(x, y)}{\partial y}\,\partial x \right] dy.$$

However, we will not use this expression to solve equations.

Example. Solve $\dfrac{dy}{dx} = \dfrac{3x(2-xy)}{x^3+2y}$.

Solution. Well,

$$(x^3+2y)\,dy = 3x(2-xy)\,dx \text{ or } 3x(xy-2)\,dx + (x^3+2y)\,dy = 0.$$

So $M(x,y) = 3x(xy-2)$ and $N(x,y) = x^3+2y$. Then

$$\frac{\partial M(x,y)}{\partial y} = 3x^2 = \frac{\partial N(x,y)}{\partial y},$$

so the DE is exact. We need

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x^2y - 6x \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3+2y.$$

So $F(x,y) = \displaystyle\int (3x^2y - 6x)\,\partial x = x^3y - 3x^2 + \varphi(y)$ where y is some function of y. Also, $\dfrac{\partial F(x,y)}{\partial y} = x^3 + \varphi'(y)$ so we need $x^3 + \varphi'(y) = N(x,y) = x^3 + 2y$. Therefore $\varphi'(y) = 2y$ and $\varphi(y) = y^2 + c_0$. Hence, $F(x,y) = x^3y - 3x^2 + y^2 + c_0$. A one parameter family of solutions is given by $F(x,y) = c_1$. So the solution is $x^3y - 3x^2 + y^2 = c$.

Note. We clarify the previous example with a theorem.

Theorem 2.2. Suppose that the DE $M(x,y)\,dx + N(x,y)\,dy = 0$ satisfies the differentiability requirements of Theorem 2.1 and is exact in a rectangular domain D. Then a *one-parameter family of solutions* is given by $F(x,y) = c$ where F satisfies

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y)$$

for all $(x,y) \in D$ and c is an arbitrary constant.

Note. We might start with an equation that is not exact and multiply it by a certain factor and "create" an exact DE. This is the idea of an integrating factor, which we now define.

Definition. If the DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is <u>not</u> exact in a domain D but the DE

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

<u>is</u> exact, then $\mu(x,y)$ is called an *integrating factor* of the original DE.

Note. Multiplying by integrating factors may (1) introduce new solutions, (2) loose solutions of the original DE, or (3) both. This means that all solutions obtained in this manner must be checked.

Example. Solve $y\,dx + 2x\,dy = 0$ where $x = 1$ when $y = 1$. HINT: The integrating factor is y.

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