Section 2.2. Separable Equations and Equations Reducible to This Form

Note. In this section we deal with (in my opinion) the easiest to solve DEs.

Definition. An equation of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0$$

is called a *separable equation.*

Note. Note that a separable equation has the integrating factor $\dfrac{1}{f(x)G(y)}$. Multiplying both sides of the equation by this, gives:

$$\frac{F(x}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0.$$

This is an exact DE since $\dfrac{\partial}{\partial y}\left[\dfrac{F(x)}{f(x)}\right] = 0 = \dfrac{\partial}{\partial x}\left[\dfrac{g(y)}{G(y)}\right]$. We can determine a one-parameter family of solutions simply by integrating:

$$\int \frac{F(x)}{f(x)}\,dx + \int \frac{g(y)}{G(y)}\,dy = c.$$

By dividing by $f(x)G(y)$, we have assumed $f(x) \neq 0$ and $G(y) \neq 0$. We will assume these quantities are not zero and carry out the integration. We will then check for solutions of the form $y =_0$ (constant) and $G(y_0) = 0$. We see that these equations will define functions that are solutions by writing the equation as:

$$f(x)g(y)\underbrace{\frac{dy}{dx}}_{=0 \text{ if } y=y_0} + F(x)\underbrace{G(y)}_{=0 \text{ if } y=y_0} = 0.$$

Example. Solve $3(y^2+1)\,dx + 2xy\,dy = 0$.

Solution. Lets use the integration factor $\dfrac{1}{x(y^2+1)}$. This gives

$$\frac{3}{x}\,dx + \frac{2y}{y^2+1}\,dy = 0.$$

Here we must assume $x \neq 0$ but there is no restriction on y. So

$$\int \frac{3}{x}\,dx + \int \frac{2y}{y^2+1}\,dy = 0 \text{ or } 3\ln|x| + \ln(y^2+1) = c_1 \text{ or}$$

$$\ln|x|^3 + \ln(y^2+1) = c_1 \text{ or } \exp(\ln|x|^2 + \ln(y^2+1)) = \exp(c_1)$$

and so $|x|^3(y^2+1) = e^{c_1}$. We can say, replacing e^{c_1} with c, that $x^3(y^2+1) = c$. (There is a problem if $x = 0$. Also, $x > 0$ and $x < 0$ are, in a sense, different cases.)

Example. Solve $\dfrac{dy}{dx} = \dfrac{xy}{\sqrt{x^2-4}}$ where $y(2) = 5$.

Solution. We have $\dfrac{1}{y}\,dy - \dfrac{x}{\sqrt{x^2-4}}\,dx = 0$. Here we have assumed that $y \neq 0$. Integrating

$$\int \frac{1}{y}\,dy - \int \frac{x}{\sqrt{x^2-4}}\,dx = 0 \text{ or } \ln|y| - \sqrt{x^2-4} = c_1 \text{ or}$$

$$\ln|y| = (x^2-4)^{1/2} + c_1 \text{ or } |y| = e^{c_1}e^{(x^2-4)^{1/2}}.$$

So $y = ce^{(x^2-4)^{1/2}}$ where c is an arbitrary constant. Notice that $y = 0$ is a solution. We get this from the one-parameter family of solutions when $c = 0$. In order to satisfy $y(2) = 5$, c must be 5 and the solution to the IVP is $y = 5e^{(x^2-4)^{1/2}}$.

Note. It is possible to change a certain other class of DEs into a separable equation. We need some definitions.

Definition. A continuous function $F(x, y)$ is said to be *homogeneous of degree n* if, for every t, $F(tx, ty) = t^n F(x, y)$.

Example. The function $F(x, y) = x^2 + xy$ is homogeneous of degree 2.

Definition. A first-order DE $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be *homogeneous* if $M(x, y)$ and $N(x, y)$ are both homogeneous of degree n.

Theorem 2.3. If $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation, then the change of variables $y = vx$ transforms the above DE into a separable DE in the variables v and x.

Example. Solve $(xy)\,dx - (2x^2 + y^2)\,dy = 0$.