Section 2.3. Linear Equations and Bernoulli Equations

Note. We have already defined when a DE is linear. In this section, we consider first order linear DEs.

Definition. A first order ODE is *linear* if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Note. We now find an integrating factor for this type of DE. We have

$$P(x)y - Q(x) + \frac{dy}{dx} = 0 \text{ or } (P(x)y - Q(x))\, dx + dy = 0.$$

Notice this is in the form of a first order linear DE $M(x,y)\,dx + N(x,y)\,dy = 0$ where $M(x,y) = P(x)y - Q(x)$ and $N(x,y) = 1$. Suppose $\mu(x)$ is an integrating factor (we'll see that we can find an integrating factor that depends only on x). Then

$$[\mu(x)P(x)y - \mu(x)A(x)]\, dx + dy = 0.$$

This is exact if

$$\frac{\partial}{\partial y}[\mu(x)P(x)y - \mu(x)Q(x)] = \frac{\partial}{\partial x}[\mu(x)]$$

or $\mu(x)P(x) = d\mu(x)/dx$ or $\mu(x)P(x) = d\mu/dx$ or $P(x)\,dx = d\mu/\mu(x)$ or $\ln|\mu(x)| = \int P(x)\,dx$ or $|\mu(x)| = e^{\int P(x)\,dx}$. If we assume $\mu > 0$ then we have $\mu(x) = e^{\int P(x)\,dx}$. Using this integrating factor, we get

$$e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y = Q(x)e^{\int P(x)\,dx} \text{ or } \frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = Q(x)e^{\int P(x)\,dx}.$$

Integrating we get

$$e^{\int P(x)\,dx} y = \int Q(x) e^{\int P(x)\,dx}\,dx + C \text{ or } y = e^{-\int P(x)\,dx}\left\{\int Q(x) e^{\int P(x)\,dx}\,dx + C\right\}.$$

In conclusion, we have the following.

Theorem 2.4. The linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ has an integrating factor $e^{\int P(x)\,dx}$. A one parameter family of solutions is

$$y = e^{-\int P(x)\,dx}\left\{\int Q(x) e^{\int P(x)\,dx}\,dx + C\right\}.$$

Example. Solve $x^2\frac{dy}{dx} + 3xy = \frac{1}{x}\cos x$.

Solution. We have $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^3}\cos x$. Then $P(x) = 3/x$ and $\int P(x)\,dx = 3\ln|x|$ (I owe you a constant of integration...). So $e^{\int P(x)\,dx} = e^{3\ln|x|} = |x|^3 = \pm x^3$. Remember, we are assuming the integrating factor is positive. (of course, if the integrating factor is negative, which occurs when $x < 0$, then we can use $-x^3$ as an integrating factor. However, since we multiply both sides of the DE by the factor, it does not matter.) So the DE becomes:

$$x^3\frac{dy}{dx} + 3x^2y = \cos x$$

or $\frac{d}{dx}[x^3y] = \cos x$ or $x^3y = \int \cos x\,dx$ or $x^3y = \sin x + C$ (there's your constant!). So $y = \frac{1}{x^3}\sin x + \frac{C}{x^3}$.

Definition. A DE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a *Bernoulli differential equation.*

Note. Notice that if $n = 0$ or 1, then a Bernoulli equation is actually a linear equation. In fact, we can transform a Bernoulli DE into a linear DE as follows.

Theorem. Suppose $n \neq 0$ and $n \neq 1$. Then the transformation $v = y^{1-n}$ reduces the Bernoulli DE $\frac{dy}{dx} + P(x)y = Q(x)y^n$into a linear equation in v. (Notice that if $v = y^{1-n}$ then $dv/dx = (1-n)y^{-n}\,dy/dx$.)

Example. Solve $x\frac{dy}{dx} + y = -2x^6y^4$.

Solution. This is a Bernoulli DE. Let $v = y^{1-4} = y^{-3}$. Then $dv/dx = -3y^{-4}\,dy/dx$. So the DE becomes:

$$x\left(\frac{y^4}{-3}\frac{dv}{dx}\right) + y = -2x^6y^4$$

$$\text{or } \frac{x}{-3}\frac{dv}{dx} + y^{-3} = -2x^6 \text{ or } \frac{x}{-3}\frac{dv}{dx} + v = -2x^6 \text{ or } \frac{dv}{dx} - \frac{3}{x}v = 6x^5.$$

This is a linear DE of the first order. So the integrating factor is $e^{\int P(x)\,dx} = e^{-\int 3/x\,dx} = x^{-3}$. Now, $x^{-3}\frac{dv}{dx} - 3x^{-4}v = 6x^2$ or $\frac{d}{dx}[x^{-3}v] = 6x^2$. So $x^{-3}v = \int 6x^2\,dx = 2x^3 + C$ or $v = 2x^6 + Cx^3$ and $y^{-3} = 2x^6 + Cx^3$ or $y = (2x^6 + Cx^3)^{-1/3}$.