Chapter 3. Applications of First-Order Equations

Section 3.2. Problems in Mechanics

Note. In this section we use Newton's Laws of Motion to introduce DEs with very physical meaning. We start by reviewing Newton's laws.

Note. Newton's Second Law of Motion states that "force = mass × acceleration." if x is the position of the particle at time t, then

$$\text{velocity} = \frac{dx}{dt} \text{ and } \text{acceleration} = \frac{d^2x}{dt^2}.$$

So Newton's Second Law becomes $F = m\frac{d^2x}{dt^2}$. If we let $v =$ velocity at time $t = dx/dt$, then it becomes $F = m\frac{dv}{dt}$. Also, if we consider velocity as a function of position x, then by the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

and so Newton's Second Law is also $F = mv\frac{dv}{dx}$. For these equations to be valid, we must be careful with the units of measure. Table 3.1 on page 79 of the book lists three systems of units. Near the surface of the Earth, the acceleration due to gravity is $g = 32 \text{ ft/sec}^2 = 9.8 \text{ m/sec}^2 = 980 \text{ cm/sec}^2$. The force an object exerts due to the downward gravitational pull is called "weight." On the Earth, $w = mg = F$. So far, in math class you have probably only considered falling bodies in the absence of air resistance. We will now consider some problems where we account for air resistance.

Example. Page 88 Number 4. A ship weighs 32,000 tons starts from rest under the force of a constant propeller thrust of 100,000 lb. The resistance in pounds is numerically equal to $8,000v$, where v is in feet per second. Find the velocity of the ship as a function of the time. Find the limiting velocity (that is, the limit of v as $t \to \infty$).

Solution. Well, $F = ma = m\,dv/dt$. The force propelling the ship is 100,000 lb and the force of resistance is $8,000v$. So the total force is $F = 100,000 - 8,000v$. The mass is

$$m = \frac{w}{g} = \frac{(32,000 \text{ tons})(2,000 \text{ lb/ton})}{32 \text{ ft/sec}^2} = 2,000,000 \text{ lb sec}^2/\text{ft}.$$

(Recall that 1 lb sec^2/ft is the unit of mass of 1 slug.) Since $F = ma$ then $F = 100,000 - 8,000v = 2,000,000\,dv/dt$. Also, $V(0) = 0$ (so this is an IVP). We get the separable DE $\dfrac{dv}{25 - 2v} = \dfrac{dt}{500}$ and $v = \dfrac{25}{2}(1 - e^{-t/250})$. Notice that

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{25}{2}(1 - e^{-t/250}) = \frac{25}{2} = 12.5 \text{ ft/sec}.$$

Note. We can also introduce friction.

Example. Page 90 Number 13. A man pushing a loaded sled across a level field of ice at the constant speed of 10 ft/sec. When the man is halfway across the ice field, he stops pushing and lets the loaded sled continue on. The combined weight of the sled and its load is 80 lb; the air resistance (in pounds) is numerically equal to $\frac{3}{4}v$, where b is the velocity of the sled (in feet per second); and the coefficient of friction of the runners on the ice is 0.04. How far will the sled continue to move after the man stops pushing?

Solution. The coefficient of friction μ gives the resistance due to the roughness of the surface; the resistance is μ times the force acting perpendicular to the surface (in this case, just the weight). So the force due to friction is (0.04)(80 lb). The total force on the sled is $-\frac{3}{4}v - (0.04)(80)$. The mass is $m = \frac{w}{g} = \frac{80 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{5}{2}$ and the acceleration is dv/dt. Since $F = ma$ then $-\frac{3}{4}v - (0.04)(80) = \frac{5}{2}\frac{dv}{dt}$. Also, $v(0) = 10$ ft/sec. Therefore we get

$$\frac{dv}{15v + 64} = -\frac{dt}{50} \text{ or } v = \frac{214}{15}e^{-3t/10} - \frac{64}{15}.$$

Integrating and assuming $x(0) = 0$, we get

$$x = -\frac{428}{9}e^{-3t/10} - \frac{64t}{15} + \frac{428}{9}.$$

Setting $v = 0$, we get $t \approx 4.02$ and at $t = 4.02$, $x \approx 16.18$ feet.

Example. Page 91 Number 18. A rocket of mass m is fired vertically upward from the surface of the Earth with initial velocity $v = v_0$. The only force on the rocket that we consider is the gravitational attraction of the Earth. Then, according to Newton's Law of Gravitation, the acceleration a of the rocket is given by $a = -k/x^2$, where $k > 0$ is a constant of proportionality and x is the distance "upward" from the center of the Earth along the line of motion. At time $t = 0$, $x = R$ (where R is the radius of the Earth), $a = -g$ (where g is the acceleration due to gravity), and $v = v_0$. Express $a = dv/dt$ as in Equation (3.23), apply the appropriate initial data, and note that v satisfies the differential equation

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}.$$

Solve this differential equation, apply the appropriate initial condition, and thus express v as a function of x. In particular, show that the minimum value of v_0 for which the rocket will escape from the Earth is $\sqrt{2gR}$. This is the so-called *velocity of escape*; and using $R = 4000$ miles, $g = 32$ ft/sec^2, one find that this is approximately 25,000 mph (or 7 mi/sec).

Solution. We have $\frac{k}{x^2} = F = m\frac{dv}{dt}$. At $t = 0$, $x = R$, $a = -g$, and $v = v_0$, so $F = \frac{k}{R^2} = m(-g)$, so that $k = -mgR^2$. Also $F = mv\frac{dv}{dx}$, so $\frac{-mgR^2}{x^2} = mv\frac{dv}{dx}$ or $\frac{-gR^2}{x^2} = v\frac{dv}{dx}$.

So we have $v\,dv = -gR^2\frac{1}{x^2}\,dx$ or $\frac{1}{2}v^2 = gR^2\frac{1}{x}+c$ for some $c \in \mathbb{R}$. With $x(0) = R$ and $v(0) = v_0$ we have $\frac{1}{2}v_0^2 = gR^2\frac{1}{R}+c$ or $c = \frac{v_0^2}{2}-gR$. Then $\frac{1}{2}v^2 = gR^2\frac{1}{x}+\frac{v_0^2}{2}-gR$, or

$$v^2 = 2gR^2\frac{1}{x} + v_0^2 - 2gR.$$

If $v_0 \geq \sqrt{2gR}$ then for $x > 0$, v is always positive and the object will escape! So the minimum escape velocity is $v_0 = \sqrt{2gR}$.

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