Section 3.3. Rate Problems

Note. In this section we consider three types of rate problems: (1) Decay Problems, (2) Growth Problems, and (3) Mixture Problems.

Note. The rate at which radioactive nuclei decay is proportional to the number of such nuclei present. With x as the mass of radioactive substance, we have the DE $\frac{dx}{dt} = -kx$ where $k > 0$. Suppose $x(0) = x_0$. Then $x = x_0 e^{-kt}$.

Definition. The *half life* of a radioactive substance is the time T it takes for the amount of substance present to reach 1/2 the original amount. So we set $\frac{1}{2}x_0 = x_0 e^{-kT}$ so that $T = \ln(2)/k$. The half life of some common radioactive elements is as follows.

Isotope	**Half Life**
Uranium, U^{238}	4.5×10^9 years
Plutonium, Pu^{230}	24,360 years
Carbon, C^{14}	5,730 years
Radium, Ra^{226}	1,620 years
Einsteinium, Es^{254}	276 days
Nobelium, No^{257}	23 seconds

Example. Carbon 14 dating was performed on the Shroud of Turin in late 1989. The organic fibers of the clothe were found to contains 89% of the C^{14} of such fibers today. How old is the shroud?

Solution. The amount of C^{14} present at a time t after the decay starts (that is, after the plant producing the fibers has died) is $x = x_0 e^{-kt}$. First, we need to find k. Since we have the half life of C^{14} above, we can set $x = x_0/2$ and $t = 5,730$ years to get $x_0/2 = x_0 e^{-k(5,730 \text{ years})}$ and we find $k \approx 0.00012097$ (when t is expressed in years). Now with the amount as $x = (0.89)x_0$ and this value of k, we can solve for age t: $0.89x_0 = x_0 e^{-(0.00012097)t}$ or $0.89 = e^{-(0.00012097)t}$ or $t \approx 963$ years. Notice that this places the estimated date of the shroud as $1989 - 963 = 1026$.

Note. We now shift form exponential decay to exponential growth and study "Growth Problems." In an ideal environment, the rate of growth of a population is proportional to the size of the population, i.e. $\dfrac{dx}{dt} = kx$ where $k > 0$ and so $x = x_1 e^{kt}$. As with half life in exponential decay, we can talk about *doubling time* T and get $T = \ln(2)/k$, as before.

Note. In the "real world," no environment is ideal. In fact, an environment will have some *carrying capacity*, K. This reflects the maximum number of individuals which an ecosystem can support. The limits are due to crowding, competition for food, mates, etc. As the size of the population nears K, the rate of growth of the population slows. This is modeled by the differential equation $\dfrac{dx}{dt} = kx(K - x)$. Notice for $x > K$ we have $dx/dt < 0$ and the population decreases. We can also express the DE as

$$\frac{dx}{dt} = kx - \lambda x2 = \lambda x \left(\frac{k}{\lambda} - x \right).$$

Growth which follows the DE is called *logistic growth*. Notice the DE is separable.

If $x(0) = x_0$ then

$$x(t) = \frac{kx_0}{\lambda x_0 + (k - \lambda x_0)e^{-kt}}.$$

Notice that $\lim_{t\to\infty} x(t) = k/\lambda$; this is the carrying capacity in terms of k and λ.

Note. We now consider the concentration of a substance (such as salt) in a tank. The tank will have the substance being added to the tank at some fixed rate, and material is removed from the tank at a fixed rate. We will assume the substance which is added to the tank is instantaneously mixed. The is an example of a "Mixture Problem."

Example. Page 105 Number 21. A tank initially contains 100 gal of brine in which there is dissolved 20 lb of salt. Starting at time $t = 0$, brine containing 3 lb of dissolved salt per gallon flows into the tank at the rate of 4 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt in the tank as a function of time.

Solution. Let x be the amount of salt at time t. Then $x_0 = 20$ lb. The salt *enters* the tank at a rate of (3 lb/gal) $\times$ (4 gal/min) $= 12$ lb/min. The salt *leaves* the tank at a rate of (x lb/gal) $\times$ (4 gal/min) $= x/25$ lb/min. So $dx/dt = 12 - x/25$ and $x(0) = 20$. We have $25\dfrac{dx}{dt} = 300 - x$ or $\dfrac{25}{300 - x}\,dx = dt$ so that $-25\ln|300 - x| = t + c$ for some c. This gives $\ln|300 - x| = -t/25 - c/25$ or (exponentiating) $300 - x = e^{-t/25}e^{-c/25} = ke^{-t/25}$ where $k = e^{-c/25}$. So we have $x(t) = 300 - ke^{-t/25}$. With $x(0) = 20$ we find $k = 280$ so that the amount of salt in the tank at time t is $x(t) = 300 - 280e^{-t/25}$.

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