Chapter 4. Explicit Methods of Solving Higher-Order Linear Differential Equations

Section 4.1. Basic Theory of Linear Differential Equations

Note. In this section we in some detail solutions of nth order linear DEs. We could call this section "Linear Algebra Meets DEs!" We will use the y' notation as opposed to dy/dx for this chapter. We start by recalling the definition of a linear DE.

Definition/Note. A linear DE of order n in the dependent variable y and the independent variable x is an equation of the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

where $a_0(x) \not\equiv 0$. We shall assume $a_1, a_1, \ldots, a_n$ and F are continuous on an interval $x \in [a, b]$ and $a_0(x) \neq 0$ for $x \in [a, b]$. The term $F(x)$ is called the *nonhomogeneous term*. If $F(x) = 0$ then the DE is called *homogeneous*. (This is the same idea we had for homogeneous before, except that we were only dealing with first order DEs before, and they were not necessarily linear.)

Note. The following is an important theorem concerning IVPs.

Theorem 4.1. Consider

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

where $a_0, a_1, \ldots, a_n$ and F are continuous for $x \in [a, b]$ and $a_0(x) \neq 0$ for $x \in [a, b]$. Let $x_0 \in [a, b]$ and let $c_0, c_1, \ldots, c_{n-1}$ be any real constants. Then there exists a unique solution of the DE such that

$$f(x_0) = c_0, f'(x_0) = c_x, \ldots f^{(n-1)}(x_0) = c_{n-1}$$

and the solution is defined over the entire interval $[a, b]$.

Corollary. If in the above theorem we have $c_0 = c_1 = \cdots = c_{n-1} = 0$ then the unique solution is $f(x) \equiv 0$.

Note. An nth order homogeneous linear DE has an important property related to linear algebra. To discuss this, we first need a definition.

Definition. If $f_1, f_2, \ldots, f_m$ are functions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is called a *linear combination* of the f_i's.

Theorem 4.2. Basic Theorem on Linear Homogeneous Differential Equations.

If $f_1, f_2, \ldots, f_m$ are each solutions of a linear homogeneous DE, then any linear combination of these functions is also a solution.

Note. We are interested in finding solutions f_i mentioned in Theorem 4.2 and in how many of these f_i there are.

Definition. The n functions $f_1, f_2, \ldots, f_n$ are called *linearly independent* on $[a, b]$ if there are some constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in [a, b]$.

Definition. If a collection of functions is not linearly dependent, it is said to be *linearly independent*. In this case, if

$$c_1 f(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in [a, b]$ then $c_1 = c_2 = \cdots = c_n = 0$.

Note. We can now state one of the most important theorems concerning linear DEs.

Theorem 4.3. The nth order homogeneous linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

always has n linearly independent solutions.

Definition. If $f_1, f_2, \ldots, f_n$ are n linearly independent solutions to an nth order homogeneous linear DE then these functions make up a *fundamental set* of solutions of the DE. The *general solution* is

$$f(x) = c_1 f_1(x) + c_2 f_2(x_) + \cdots + c_n f_n(x)$$

where the c_i's are arbitrary constants.

Example. The 2nd order homogeneous linear DE $y'' + y = 0$ has as solutions the two linearly independent functions $f_z(x) = \sin x$ and $f_2(x) = \cos x$. So $\{\sin x, \cos x\}$ is a fundamental set of solutions and the general solution is

$$f(x) = c_1 \sin x + c_2 \cos x$$

where c_1 and c_2 are arbitrary.

Note. There is a convenient way to check for linear independence of several functions. We will use the following.

Definition. Let $f_1, f_2, \ldots, f_n$ be n real functions each of which has an $(n-1)$th derivative on the interval $[a, b]$. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$$

is called the *Wronskian* of the n functions. Notice that the Wronskian is itself a function of x.

Theorem 4.4, 4.5. The Wronskian of the n functions $f_1, f-2, \ldots, f_n$ above is either identically zero on $[a, b]$ or else is never zero on $[a, b]$. The functions are linearly independent if and only if the determinant is nonzero.

Note. Recall that determinants can be calculated for 2×2 and 3×3 matrices as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Note. We can use the Wronskian to show that, in fact, $\sin x$ and $\cos x$ are linearly independent:

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = (\sin x)(-\sin x) - (\cos x)(\cos x) = -(\sin^2 x + \cos^2 x) = -1 \neq 0.$$

Note. Much like knowing a certain zero of a polynomial allows you to factor the polynomial into a linear factor and a new polynomial of degree one less than the original polynomial, if we know one solution of an nth order linear homogeneous linear DE then we can reduce the order of the DE to an $(n-1)$ order DE.

Theorem 4.6. Let f be a nontrivial solution of the nth order homogeneous linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

then the transformation $y = f(x)v$ reduces the DE to an $(n-1)$ order homogeneous linear DE in the dependent variable $x = dv/dw$.

Example. We illustrate Theorem 4.6 for $n = 2$. Suppose f is a known solution of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Let $y = f(x)v$. Then $y' = f(x)v' + f'(x)v$ and $y'' = f(x)v'' + 2f'(x)v' + f'(x)v$. So the DE becomes

$$a_0(x)(f(x)v'' + 2f'(x)v' + f'(x)v) + a_1(x)(f(x)v' + f'(x)v) + a_2(x)f(x) = 0$$

or

$$a_0(x)f(x)v'' + [2a_0(x)f'(x) + a_1(x)f(x)]v' + [a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v = 0$$

or $a_0(x)f(x)v'' + [2a_0(x)f'(x) + a_1(x)f(x)]v' = 0$. Letting $w = v'$ gives $dw/dx = v''$ and

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0.$$

This is a homogeneous linear DE of the first order. It is also separable. Solving, we get

$$w = \frac{c\exp\left[-\int a_1(x)/a_0(x)\,dx\right]}{[f(x)]^2}$$

and

$$v = \int \frac{c\exp\left[-\int a_1(x)/a_0(x)\,dx\right]}{[f(x)]^2}\,dx$$

so

$$y = f(x)\int \frac{c\exp\left[-\int a_1(x)/a_0(x)\,dx\right]}{[f(x)]^2}\,dx = g(x).$$

We can show (see page 126) that $g(x)$ and $f(x)$ are linearly independent. So the general solution of the above DE is $c_1f(x) + c_2g(x)$. We summarize this in the following theorem.

Theorem 4.7. Let f be a nontrivial solution of the second order homogeneous linear DE

$$a_0(x)y'' + a_1(x)y' + a_2(x) = 0.$$

Then the transformation $y = f(x)v$ reduces this DE to the first order linear homogeneous DE

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

in the dependent variable w, where $w = v'$, which has the solution

$$w = \frac{e^{-\int a_1(x)/a_0(x)\,dx}}{[f(x)]^2}.$$

So the other solution to the original DE is

$$g(x) = f(x)v(x) = f(x)\int w\,dx = f(x)\int \frac{e^{-\int a_1(x)/a_0(x)\,dx}}{[f(x)]^2}\,dx.$$

The general solution to the original DE is $c_1f(x) + c_2g(x)$ where c_1 and c_2 are arbitrary constants.

Example. Use the fact that $f(x) = x$ is a solution of $2x^2y'' + xy' - y = 0$ to find the general solution.

Solution. Let $y = xv$. Then $y' = xv' + v$ and $y'' = xv'' + 2v'$. So the DE becomes

$$2x^2(xv'' + 2v') + x(xv' + v) - (xv) = 0$$

or $(2x^3)v'' + (5x^2)v' = 0$. Let $w = v'$ then we have $(2x^2)w' + (5x^2)w = 0$ or

$$2x^3\frac{dw}{dx} + (5x^2)w = 0 \text{ or } \frac{1}{w}\,dw + \frac{5}{2x}\,dx = 0 \text{ or } \ln|w| + \frac{5}{2}\ln|x| = c_0,$$

so $|w||x|^{5/2} = e^{c_0}$ or $|w| = e^{c_0}|x|^{-5/2}$ or $w = c_1x^{-5/2}$. Then $v = \int w\,dx = \int c_1x^{-5/2}\,dx = c_2x^{-3/2} + c_3$. Then $g(x) = vx = c_2x^{-1/2} + c_3x$. Notice that we could take all the above constants to be 0 (or anything convenient). The general solution is

$$k_1 f(x) + k_2 g(x) = k_1 x + k_2 x^{-1/2}.$$

Note. We now consider ways to apply the methods of solving homogeneous DEs to nonhomogeneous DEs.

Theorem 4.8. Let v be any solution of the nonhomogeneous DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

and let u be any solution of the nonhomogeneous DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0.$$

Then $u + v$ is also a solution of the above nonhomogeneous DE.

Note. This result allows us to talk about the general solution of an nth order nonhomogeneous linear DE.

Theorem 4.9. Let y_p be a particular solution to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x).$$

Let y_c be a general solution of

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0.$$

Then every solution of the nonhomogeneous DE is of the form $y_p + y_c$ for arbitrary constants $c_1, c_2, \ldots, c_n$ is y_c.

Definition. For the DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

the general solution y_c of the associated homogeneous DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

is called the *complementary function* of the nonhomogeneous DE. Any particular solution of the nonhomogeneous DE is called a *particular integral* of this DE. The solution $y_c + y_p$ is the *general solution* of the nonhomogeneous DE.

Example. Given that $y = e^x$ is a particular integral of $y'' + y = 2e^x$, find the general solution.

Solution. Recall that the general solution of $y'' + y = 0$ is $y_c = c_2 \sin x + c_2 \cos x$. So this is the complementary function of the given nonhomogeneous DE. So the general solution of the nonhomogeneous DE is

$$\varphi(x) = c_1 \sin x + c_2 \cos x + e^x.$$

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