Section 4.2. The Homogeneous Linear Equation with Constant Coefficients

Note. In this section we explore the DE

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

where $a_0, a_1, \ldots, a_n$ are real constants.

Definition/Note. Suppose $y = e^{mx}$ is a solution of the DE above. We have $y' = me^{mx}$, $y'' = m^2 e^{mx}$, $\ldots$, $y^{(n)} = m^n e^{mx}$, so the DE becomes

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$e^{mx}(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0.$$

This is called the *auxiliary equation* or *characteristic equation* of the DE.

Note. Recall that there are three possibilities for the roots of a polynomial with real coefficients:

1. They may be real and distinct,

2. they may be real and repeated, or

3. they may occur in complex conjugate pairs.

We now consider these cases in order. In the first case, we have the following result.

Theorem 4.11. Consider $a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny = 0$. If the auxiliary equation

$$a_0m^n + a_1^{n-1} + \cdots + a_n = 0$$

has n distinct real roots $m_1, m_2, \ldots, m_n$ then the general solution of the DE is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Note. This theorem implies that the powers of e^x are linearly independent.

Example. Page 143 Number 2. Find the general solution of $y'' - 2y' - 3y = 0$.

Solution. The auxiliary equation is $m^2 - 2m - 3 = 0$ or $(m - 3)(m + 1) = 0$. So the auxiliary equation has has distinct real roots $m = -1$ and $m = 3$. So the general solution is $y = c_1e^{-x} + c_2e^{3x}$. Notice that

$$\begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = (e^{-x})(3e^{3x}) - (e^{3x})(-e^{-x}) = 3e^{2x} + e^{2x} = 4e^{2x} \neq 0.$$

So e^{-x} and e^{3x} are in fact linearly independent.

Note. In the case that the auxiliary equation has repeated real roots, we have the following.

Theorem 4.12. Suppose the nth order homogeneous linear DE with constant coefficients has an auxiliary equation with repeated real roots. Suppose the roots are $m_1, m_2, \ldots, m_k$ where m_i is a root of multiplicity j_i for $i = 1, 2, \ldots, k$. Then notice $j_1 + h_2 + \cdots + j_k = n$. The general solution of the DE is

$$y = (c_1 + c_2 x + \cdots + c_{j_1} x^{j_1 - 1})e^{m_1 x} + (c_{j_1+1} + c_{j_1+2} x + \cdots + c_{j_2} x^{j_2-1})e^{m_2 x}) + \cdots$$

$$+(c_{j_{k-1}+1} + c_{j_{k-1}+2} x + \cdots + c_{j_k} x^{j_k - 1})e^{m_k x}).$$

Note. If, for example, 5 is a root of the auxiliary equation of multiplicity 3, then $(c_1 + c_2 x + c_3 x^2)e^{5x}$ is part of the general solution of the associated homogeneous linear DE with constant coefficients.

Example. Page 143 Number 14. Find the general solution of $4y''' + 4y'' - 7y' + 2y = 0$.

Solution. The auxiliary equation is $4m^3 + 4m^2 - 7m + 2 = 0$. To find the roots, we apply the Rational Root Theorem (see page 554) and test the following as possible roots: $\pm 2/4, \pm 2/2, \pm 2/1$. We find that $m = -2$ is a root and we have $4m^3 + 4m^2 - 7m + 2 = (m + 2)(4m^2 - 4m + 1) = (m + 2)(2m - 1)^2 = 0$. So the roots are $m = -2, 1/2, 1/2$ and the general solution is $y = c_1 e^{-2x} + (c_2 + c_3 x)e^{x/2}$.

Note. In the case that the auxiliary equation has roots, which appear in complex conjugate pairs, then from the identity (called "Euler's formula") $e^{i\theta} = \cos\theta + i \sin\theta$ we get the following theorem.

Theorem 4.13. If the auxiliary equation of an nth order homogeneous linear DE with constant coefficients has the complex roots $a+bi$ and $a-bi$, neither repeated, then the corresponding part of the general solution of the DE is

$$y = e^{ax}(c_1 \sin(bx) + c_2 \cos(bx)).$$

If $a+bi$ and $a-bi$ are each k-fold roots then the corresponding part of the general solution is

$$y = e^{ax}[(c_1+c_2x+c_2x^2+\cdots+c_kx^{k-1})\sin(bx)+(d_1+d_2x+d_3x^2+\cdots+d_kx^{k-1})\cos(bx)].$$

Revised: 2/16/2019