Section 4.3. The Method of Undetermined Coefficients

Note. In this section we deal with the nonhomogeneous linear DE with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)$$

where $a_0, a_1, \ldots, a_n$ are real constants and $F(x)$ is a function of x.

Example. Find the general solution of $y'' - 2y' - 3y = 2e^{4x}$.

Solution. The associated homogeneous DE is $y'' - 2y' - 3y = 0$. The general solution of this homogeneous DE is $y_c = c_1 e^{-x} + c_2 e^{3x}$. Recall that this is the complementary function of the original DE. For a particular solution of the original nonhomogeneous DE, lets try $y = Ae^{4x}$. Substituting in to the original DE we find that $A = 2/5$ works so that $y_p = \frac{2}{5} e^{4x}$ is a particular integral. So the general solution of the DE is

$$y = \frac{2}{5} e^{4x} + c_1 e^{-x} + c_2 e^{3x}.$$

Example. Find the general solution of the DE: $y'' - 2y' - 3y = 2e^{3x}$.

Solution. The complementary function is (as above) $y = c_1 e^{-x} + c_2 e^{3x}$. If we try for a particular solution of the nonhomogeneous DE of the form $y = Ae^{3x}$. It *doesn't work*!!! We need more knowledge.

Definition. A function is a *UC function* if it is either one of the following:

1. x^n where n is a nonnegative integer,

2. e^{ax} where $a \neq 0$,

3. $\sin(bx + c)$ where $b \neq 0$,

4. $\cos(bx + c)$ where $b \neq 0$,

or a finite product of two or more functions of these four types.

Definition. Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions, of which the successive derivatives of f are either constant multiples or linear combinations, will be called the *UC set of f* (or the *differential family* of f).

Example. Page 151 Example 4.33. Consider $f(x) = x^2 \sin x$. Function f is a product of two UC functions and so is itself a UC function. We now find the UC set of f. We have

$$\begin{aligned} f'(x) &= 2x\sin x + x^2\cos x \\ f''(x) &= 2\sin x + 4x\cos x - x^2\sin x \\ f'''(x) &= 6\cos x - 6x\sin x - x^2\cos x \end{aligned}$$

and we see that the UC set of f is $S = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$. Notice that the UC set of $f_1(x) = x^2$ is $S_1 = \{x^2, x, 1\}$ and the UC set of $f_2(x) =$

$\sin x$ is $S_2 = \{\sin x, \cos x\}$. In fact, any element of S is a product of an element of S_1 and an element of S_2. This foreshadows the following.

Theorem. Suppose h is a UC function defined as the product fg of two basic UC functions f and g (that is, f and g are of the four types described above). Then the UC set of h is the set of all products obtained by multiplying the various members of the UC set of f by the various members of the UC set of g.

Example. We can easily find the UC set of $f(x) = x^3 e^x$. The UC set for x^3 is $S_1 = \{x^3, x^2, x, 1\}$ and the UC set for e^x is $S_2 = \{e^x\}$. So by the previous theorem, the UC set of f is $S = \{x^3 e^x, x^2 e^x, xe^x, e^x\}$.

Note. We now give an outline of the *Method of Undetermined Coefficients* and then illustrate it with several examples.

Note. The Method of Undetermined Coefficients is as follows. Consider the DE

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)$$

where $F(x)$ is a finite linear combination $F = A_1 u_1 + A_2 u_2 + \cdots + A_m u_m$ of UC functions $u_1, u_2, \ldots, u_m$. Suppose that y_c is the complementary solution of the DE.

1. For each UC function $u_1, u_2, \ldots, u_m$, form the corresponding UC sets $S_1, S_2, \ldots, S_m$.

2. Suppose that $S_j \subset S_k$. Then omit S_j from the collection of S's.

3. Suppose that S_ℓ includes one or more members which is a solution to the corresponding homogeneous DE. Then multiply each member of S_ℓ by the lowest (positive integer) power of x which produces in S_ℓ a new collection of functions none of which are a solution to the associated homogeneous DE.

4. Form a linear combination with all of the members of the S's. We now need to *determine the coefficients* of this linear combination.

5. Determine the coefficients by substituting into the original DE and equating similar terms.

Example. Lets now return to $y'' - 2y' - 3y = 2e^{3x}$. The nonhomogeneous term $2e^{3x}$ is a linear combination of the UC function e^{3x}. So let $S = \{e^{3x}\}$. However, by Step 3, e^{3x} is a solution to the associated homogeneous DE. So by Step 3, replace e^{3x} with xe^{3x}. And so $S = \{xe^{3x}$. Notice this is not a solution of the associated homogeneous DE. So let $y_p = Axe^{3x}$. The $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 3Ae^{3x}+3Ae^{3x}+9Axe^{3x} = 6Ae^{3x}+9Axe^{3x}$, so we substitute into the nonhomogeneous DE to get $(6Ae^{3x} + 9Axe^{3x}) - 2(Ae^{3x} + 3Axe^{3x}) - 3(Axe^{3x}) = 4Ae^{3x} = 2e^{3x}$ so that $A = 1/2$ and $y_p = \dfrac{1}{2}e^{3x}$. So the general solution to the nonhomogeneous DE is

$$y = \frac{1}{2}e^{3x} + c_1e^{-x} + c_2e^{3x}.$$

Example. Page 160 Number 11. Find the general solution of $y'' + 4y = 4\sin(2x) + 8\cos(2x)$.

Solution. The associated homogeneous DE is $y'' + 4y = 0$ and has general solution $y_c = c_1 \sin(2x) + c_2 \sin(2x)$. The nonhomogeneous term is $F(x) = 4\sin(2x) + 8\cos(2x)$. So $F(x)$ is a linear combination of the UC functions $\sin(2x)$ and $\cos(2x)$. Each has the UC set $S = \{\sin(2x), \cos(2x)\}$. From Step 3, $\sin(2x)$ and $\cos(2x)$ are both solutions of the associated DE so replace these with $S = \{x\sin(2x), x\cos(2x)\}$. From Step 4, suppose $y_p = Ax\sin(2x) + Bx\cos(2x)$. Substituting into the nonhomogeneous DE we find that $A = 2$ and $B = -1$, so $y_p = 2x\sin(2x) - x\cos(2x)$. Then the general solution is

$$y = y_p + y_c = 2x\sin(2x) - x\cos(2x) + c_1 \sin(2x) + c_2 \cos(2x).$$

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