Section 4.4. Variation of Parameters

Note. In this section we study a method which can be applied to a much larger class of functions than the method of undetermined coefficients.

Note. Consider the second-order DE:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x).$$

Suppose $y_c = c_1y_1(x) + c_2y_2(x)$ is the complimentary function for this DE. We now try to find a particular integral of the form $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$. We now need to find the parameters $v_1(x)$ and $v_2(x)$. This method is called *variation of parameters*. We have the <u>one</u> condition that y_p be a particular integral and we have the <u>two</u> functions with which to do this. We will take advantage of this to put a second restriction on $v_1(x)$ and $v_2(x)$. We have

$$y_p' = v_1(x)y_1'(x) + v_1'(x)y_1(x) + v_2(x)y_2'(x) + v_2(x)y_2(x).$$

We now make the second restriction

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0. \qquad (*)$$

Then

$$y_p' = v_1(x)y_1'(x) + v_2(x)y_2'(x)$$

and

$$y_p'' = v_1(x)y_1''(x) + v_1'(x)y_1'(x) + v_2(x)y_2''(x) + v_2'(x)y_2'(x).$$

Substituting into the DE we get:

$$a_0(x)[v_1(x)y_1''(x) + v_1'(x)y_1'(x) + v_2(x)y_2''(x) + v_2'(x)y_2'(x)]$$

$$+a_1(x)[v_1(x)y_1'(x) + v_2(x)y_2'(x)] + a_2(x)[v_1(x)y_1(x) + v_2(x)y_2(x)] = F(x).$$

Rearranging we have

$$v_1(x)[a_0(x)y_1''(x)+a_1(x)y_1'(x)+a_2(x)y_1(x)]+v_2(x)[a_0(x)y_2''(x)+a_1(x)y_2'(x)+a_2(x)y_2(x)]$$

$$+a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = F(x),$$

or, since the term in square brackets multiplied by $v_1(x)$ is 0 (because $y_1(x)$ is a solution to the DE) and the term in square brackets multiplied by $v_2(x)$ is 0 (because $y_2(x)$ is a solution to the DE), then this simplifies to:

$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{F(x)}{a_0(x)}. \qquad (**)$$

The two equations $(*)$ and $(**)$ form a system of two equations in two unknowns $v_1(x)$ and $v_2(x)$. Solving this system using Cramer's Rule we find:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ F(x)/a_0(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -F(x)y_2(x)a_0(x)W[y_1(x), y_2(x)]$$

$$v_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & F(x)/a_0(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = F(x)y_1(x)a_0(x)W[y_1(x), y_2(x)]$$

(notice the denominators are not zero because $y_1(x)$ and $y_2(x)$ are linearly independent). We can integrate to get

$$v_1(x) = -\int_0^x \frac{F(t)y_2(t)}{a_0(t)W[y_1(t), y_2(t)]}\,dt \text{ and } v_2(x) = -\int_0^x \frac{F(t)y_1(t)}{a_0(t)W[y_1(t), y_2(t)]}\,dt.$$

Example. Page 170 Number 6. Find the general solution of $y'' + y = \tan x \sec x$.

Solution. The complementary functions is $y_c = c_1 \cos x + c_2 \sin x$. Suppose that it has the particular integral $y_p = v_1(x)\cos x + v_2(x)\sin x$. Then $y_p' = v_1(x)\cos x - v_1(x)\sin x + v_2'(x)\sin x + v_2(x)\cos x$. We impose the condition

$$v_1'(x)\cos x + v_2'(x)\sin x = 0. \qquad (*)$$

Then $y_p' = -v_1(x)\sin x + v_2(x)\cos x$ and $y_p'' = -v_1'(x)\sin x - v_1(x)\cos x + v_2'(x)\cos x - v_2(x)\sin x$. From the given DE we then have

$$[-v_1'(x)\sin x - v_1(x)\cos x + v_2'(x)\cos x - v_2(x)\sin x]$$

$$+[v_1(x)\cos x + v_2(x)\sin x] = \tan x \sec x$$

or

$$-v_1'(x)\sin x + v_2'(x)\cos x = \tan x \sec x. \qquad (**)$$

Combining $(*)$ and $(**)$,

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\tan^2 x$$

$$v_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \tan x.$$

So

$$v_1(x) = -\int \tan^2 x\,dx = -\int (\sec^2 x - 1)\,dx = -\tan x + x + c_1,$$

$$v_2(x) = \int \tan x\,dx = \ln|\sec x| + c_2,$$

and $y_p = (-\tan x + x + c_1)\cos x + (\ln|\sec x| + c_2)\sin x$ and the general solution is

$$y = k_1 \cos x + k_2 \sin x + (-\tan x + x)\cos x + \ln|\sec x| \sin x.$$

Example. Page 170 Number 14. Find the general solution of

$$y'' + 3y' + 2y = \frac{1}{1 + e^{2x}}.$$

Solution. The complementary functions is $y_c = c_1 e^{-2x} + c_2 e^{-x}$. So we look for a particular integral of the form

$$y_p = v_1(x)e^{-2x} + v_2(x)e^{-x}.$$

Then

$$y_p' = v_1'(x)e^{-2x} + 2v_1(x)e^{-2x} + v_2'(x)e^{-x} - v_2(x)e^{-x}.$$

We impose the condition

$$v_1'(x)e^{-2n} + v_2'(x)e^{-x} = 0. \qquad (*)$$

Then $y_p = -2v_1(x)e^{-2x} - v_2(x)e^{-x}$. Also

$$y_p'' = -2v_1'(x)e^{-2x} + 4v_1(x)e^{-2x} - v_2'(x)e^{-x} + v_2(x)e^{-x}.$$

From the given DE we then have

$$v_1(x)[(4e^{-2x}) + 3(-2e^{-2x}) + 2e^{-2x})] + v_2(x)[(e^{-x} + 3(-e^{-x}) + 2(e^{-x})]$$

$$+[v_1'(x)(-2x^{-2x}) + v_2'(x)(-e^{-x})] = \frac{1}{1+e^{2x}}$$

or

$$v_1'(x)(2e^{-2x}) + v_2'(x)e^{-x} = \frac{-1}{1+e^{2x}}. \qquad (**)$$

Combining $(*)$ and $(**)$,

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-x} \\ 1/(1+e^{2x}) & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}} = \frac{-e^{-x}/(1+e^{2x})}{e^{-3x}} = \frac{-e^{2x}}{1+e^{2x}}$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 1/(1+e^{2x}) \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}} = \frac{e^{-2x}/(1+e^{2x})}{e^{-3x}} = \frac{e^{x}}{1+e^{2x}}.$$

So $v_1(x) = -\frac{1}{2}\ln(1+e^{2x}) + c_3$, $v_2(x) = \tan^{-1}(e^x) + c_4$, and

$$y_p = e^{-2x}\left(-\frac{1}{2}\ln(1+e^{2x}) + c_3\right) + e^{-x}\left(\tan^{-1}(e^x) + c_4\right).$$

Therefore the general solution is

$$y = -\frac{1}{2}e^{-2x}\ln(1+e^{2x}) + e^{-x}\tan^{-1}(e^x) + k_1e^{-2x} + k_2e^{-x}.$$

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