Section 4.5. The Cauchy-Euler Equations

Note. In Section 4.3 we dealt with linear DEs with constant coefficients. In Section 4.4 we dealt with linear DEs where the coefficients were not necessarily constant. However, we still needed the complimentary function, which we only know how to find for linear DEs with constant coefficients. In this section we deal with a certain class of linear DEs with nonconstant coefficients. We will transform this type of DE into a linear DE with constant coefficients. We can solve the new DE by the methods of Sections 4.3 and 4.4.

Definition. A linear differential equation of the form

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = F(x)$$

is a *Cauchy-Euler equation* or *equidimensional equation.*

Note. These types of equations can be solved using the technique described in the following theorem.

Theorem 4.14. The transformation $x = e^t$ reduces the Cauchy-Euler equation to a linear DE with constant coefficients.

Note. By making the transformation $x = e^t$ we are assuming $x > 0$. Unless otherwise stated, this will be the assumption. If $x < 0$ then the transformation $x = -e^t$ should be used. Notice that if $x = 0$, the Cauchy-Euler equation is even a DE!

Example. Page 176 Number 17. Find the general solution of $x^2y'' + 4xy' + 2y = 4\ln x$ where $x > 0$.

Solution. Let $x = e^t$ and $t = \ln x$. In general, by the Chain Rule $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dt}\frac{dt}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dt}\right]\frac{dt}{dx} + \frac{dy}{dt}\frac{d^2t}{dx^2}$$

$$= \left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right)\frac{dt}{dx} + \frac{dy}{dt}\frac{d^2t}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}.$$

So here we have $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dx}\frac{1}{x}$ and

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{1}{x}\right)^2 + \frac{dy}{dt}\left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

The DE then transforms to

$$x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + 4x\left(\frac{1}{x}\frac{dy}{dt}\right) + 2y = 4\ln(e^t) \text{ or } \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4t.$$

The auxiliary equation for *this* DE is $m^2 + 3m + 2 = (m+2)(m+1) = 0$. So the complementary function for this DE is $y_c = c_1e^{-2t} + c_2e^{-t}$. From the method of undetermined coefficients, we try to find a solution of the form $y_p = At + B$. We find that $y_p = \frac{1}{2}t - \frac{3}{4}$. So the general solution of the *transformed* DE is

$$y = c_1e^{-2t} + c_2e^{-t} + \frac{1}{2}t - \frac{3}{4}$$

and the general solution of the original DE is

$$y = c_1e^{-2\ln x} + c_2e^{-\ln x} + \frac{1}{2}\ln x - \frac{3}{4} = c_1\frac{1}{x^2} + c_2\frac{1}{x} + \frac{1}{2}\ln x - \frac{3}{4}.$$

Revised: 2/25/2019