Chapter 5. Applications of Second-Order Linear Differential Equations with Constant Coefficients

Section 5.1. The Differential Equation of the Vibrations of a Mass on a Spring

Note. In this section we derive the DE describing the position of a mass on a spring as a function of time. We consider the force of gravity, the restoring force of the spring, the resisting force of the medium, and other external forces.

Note. Suppose a spring has a natural length of L and a mass of size m stretches the spring to a length of $L + \ell$. Let x be the distance the spring spring is stretched beyond this equilibrium length $L + \ell$ (so x would be negative if the spring is compressed).

5.1 THE VIBRATIONS OF A MASS ON A SPRING

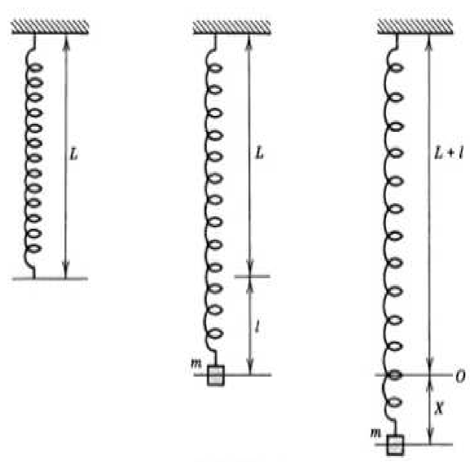

FIGURE 5.1

Then by Hooke's Law, the *restoring force* of the spring is $F_1 = -k(x + \ell)$ where k is the spring constant. The *force due to gravity* is $F_2 = mg$. So when $x = 0$, $F_1 = -F_2$ and $-k\ell = -mg$. So, $F_1 = -k(\ell + x) = -kx - mg$. We also assume a *resisting force* due to the medium (air, water, etc). This is also called the *damping force*. We assume this force is proportional to the speed of the mass and in the direction opposite the velocity. So $F_3 = -a\,dx/dt$, $a > 0$. Parameter a is called the *damping constant*. We represent any *external impressed forces* by $F_4 = F(t)$. By Newton's Second Law, $F = m \times$ acceleration, we have

$$m\frac{d^2x}{dt^2} = F_1 + F_2 + F_3 + F_4 = (-kx - mg) + mg - a\frac{dx}{dt} + F(t)$$

or

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F(t)$$

or

$$mx'' + ax' + kx = F(t).$$

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