Section 5.2. Free, Undamped Motion

Note. In this section we continue studying vibrations of a mass on a spring, but we consider *undamped motion* (i.e., $a = 0$) which is *free* in the sense that there is no external forces, i.e. $F(t) = 0$. This will lead us to simple harmonic motion.

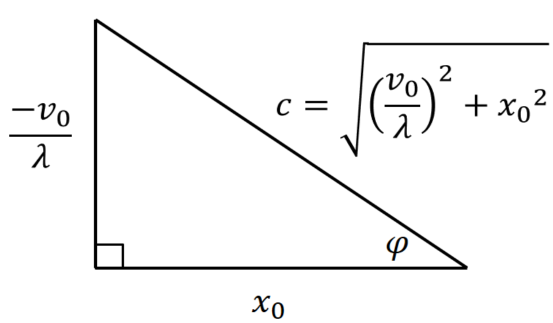
Note. The DE governing this type of motion is $mx'' + \lambda x = 0$. If we let $k/m = \lambda^2$ then this becomes $x'' + \lambda^2 x = 0$ which has auxiliary equation $M^2 + \lambda^2 = 0$, and $M = \pm\lambda i$. So the general solution is

$$x = c_1 \sin \lambda t + c_2 \cos \lambda t.$$

Now if we let $x(0) = x_0$ and $x'(0) = v_0$, then we get

$$x = \frac{v_0}{\lambda} \sin \lambda t + x_0 \cos \lambda t.$$

Note. We now introduce parameter φ as follows:



Then $\sin \varphi = \dfrac{-v_0/\lambda}{c}$ and $\cos \varphi = x_0/c$. So we can write

$$x = c\left(\frac{v_0/\lambda}{c} \sin \lambda t + \frac{x_0}{c} \cos \lambda t\right) = c\,(\cos \varphi \cos \lambda t - \sin \varphi \sin \lambda r)$$

$$= c\cos(\lambda t + \varphi) = c\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right).$$

So, for free undamped motion, we have

$$x = c\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right) \text{ where } c = \sqrt{\left(\frac{v_0}{\lambda}\right)^2 + x_0^2}.$$

Definition. The motion described here is *simple harmonic motion*. The parameter c is the *amplitude* of this motion and the *period* is $t = \dfrac{2\pi}{\sqrt{k/m}} = \dfrac{2\pi}{\lambda}$. The *frequency* is $\dfrac{\lambda}{2\pi}$. Parameter φ is the *phase angle*. (Notice that for $t = 0$, $x_0 = c\cos\varphi$.)

Note. The graph of simple harmonic motions is:

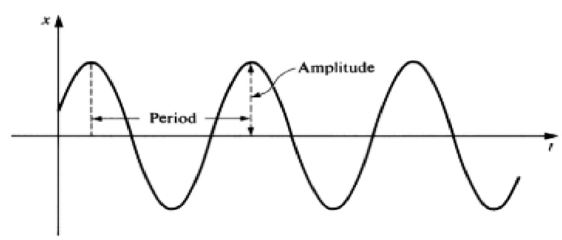


FIGURE 5.2

Example. Page 197 Number 2(a). A 16-lb weight is placed upon the lower end of a coil spring suspended vertically from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. Determine the resulting displacement as a function of time if the weight is then pulled down 4 in. below its equilibrium position and released at $t = 0$ with an initial velocity of 2 ft/sec, directed downward.

Solution. We have $mg = 16$ lb, so that $m = 16/g = 16/32 \text{ lb sec}^2/\text{ft} = 1/2$ slug. For the spring constant, with $\ell = 6$ in $= 1/2$ ft we have $mg = k\ell$ or 16 lb $= k(1/2 \text{ ft})$ or $k = 32$ lb/ft. We are given the initial conditions $x_0 = 1/3$ ft and $v_0 = 2$ ft/sec. Also,

$$\lambda = \sqrt{\frac{k}{m}} = \sqrt{\frac{32 \text{ lb/ft}}{1/2 \text{ lb sec}^2/\text{ft}}} = 8/\text{sec},$$

$$c = \sqrt{\left(\frac{v_0}{\lambda}\right)^2 + x_0^2} = \sqrt{\left(\frac{2 \text{ ft/sec}}{8/\text{sec}}\right)^2 + \left(\frac{1}{3} \text{ ft}\right)^2} = \sqrt{\frac{1}{16} + \frac{1}{9}} \text{ ft} = \sqrt{\frac{25}{144}} \text{ ft} = \frac{5}{12} \text{ ft},$$

and

$$\varphi = \cos^{-1}\left(\frac{x_0}{c}\right) = \frac{1/3}{5/12} = \frac{4}{5}.$$

Therefore,

$$x = c\cos(\lambda t + \varphi) = \frac{5}{12}\cos\left(8t + \cos^{-1}\left(\frac{4}{5}\right)\right)$$

where t is in seconds.

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