Section 5.3. Free, Damped Motion

Note. We now add the resistance of the medium to the simple harmonic motion of the previous section. Such motion of a mass on a spring is called *free, damped motion*. The DE of Section 5.1 becomes

$$mx'' + ax' + kx = 0$$

where a is the damping coefficient.

Note. We can rewrite the DE as

$$x'' + \frac{a}{m}x' + \frac{k}{m}x = 0.$$

Letting $\lambda^2 = k/m$ and $2b = a/m$, we get $x'' + 2bx' + \lambda^2 x = 0$. The auxiliary equation is $m^2 + 2bm + \lambda^2 = 0$ and the roots are

$$m = \frac{-2b \pm \sqrt{4b^2 - 4\lambda^2}}{2} = -b \pm \sqrt{b^2 - \lambda^2}.$$

Recall that we are concerned with whether the two roots are real, distinct (and real), or in a complex conjugate pair. This depends on the sign of $b^2 - \lambda^2$. This leads us to three cases.

Note. If $b < \lambda$, then we get *damped, oscillatory motion* (or *underdamped motion*). The roots of the auxiliary equation are

$$-b + \sqrt{\lambda^2 - b^2}\, i \text{ and } -b - \sqrt{\lambda^2 - b^2}\, i$$

and the solution to the DE is

$$x = e^{-bt}\left(c_1 \sin\sqrt{\lambda^2 - b^2}t + c_2 \cos\sqrt{\lambda^2 - b^2}t\right).$$

As in Section 5.2, we may write this as

$$x = ce^{-bt}\cos\left(\sqrt{\lambda^2 - b^2}t + \varphi\right)$$

where $c = \sqrt{c_1^2 + c_2^2}$, $\sin\varphi = -c_1/\sqrt{c_1^2 + c_2^2}$, and $\cos\varphi = c_2/\sqrt{c_1^2 + c_2^2}$. The factor ce^{-bt} is called the *damping factor* or *time varying amplitude.* Notice that as time increases, this term approaches 0. This motion is not periodic, but the time between successive positive maxima of x is called the *quasi period* and is $2\pi/\sqrt{\lambda^2 - b^2}$. The graph is:

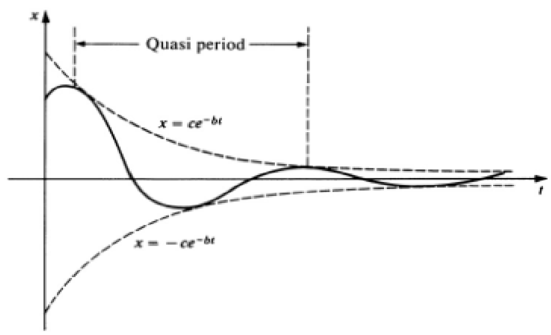

FIGURE 5.5

The solution to the DE can be written

$$x = ce^{-(a/(2m))t}\cos\left(\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}t + \varphi\right).$$

Notice that the frequency of the trig function is $\dfrac{1}{2\pi}\sqrt{\dfrac{k}{m} - \dfrac{a^2}{4m^2}}$.

Note. If $b = \lambda$ then we get *critical damping.* The roots of the auxiliary equation are $-b$ (repeated). So the general solution of the DE is

$$x = (c_1 + c_2)e^{-bt}.$$

In this case, the damping just exactly balances the force of the spring. This motion is said to be *critically damped* and is not oscillatory. The graph is this motion looks like one to the following, depending on $x'(0)$ and a:

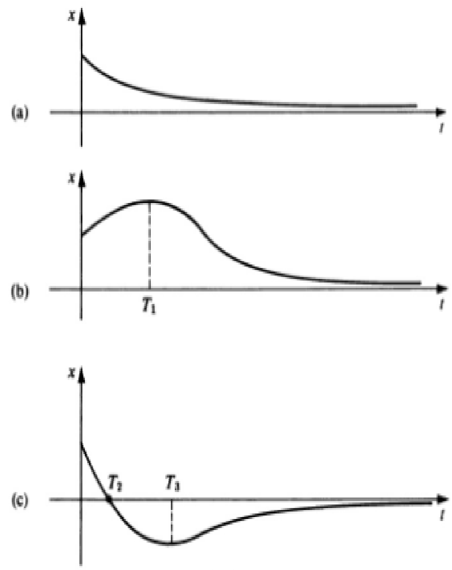

FIGURE 5.6

In this case, any decrease in a will produce oscillatory motion.

Note. If $b > \lambda$ then we get *overcritical damping.* The roots of the auxiliary equation are the distinct real numbers

$$-b + \sqrt{b^2 - \lambda^2} \text{ and } -b - \sqrt{b^2 - \lambda^2}$$

and the solution to the DE is

$$x = c_1 e^{(-b+\sqrt{b^2-\lambda^2})t} + c_2 e^{(-b-\sqrt{b^2+\lambda^2})t}.$$

The graphs of the case of critical damping apply here as well, the difference is that "small" changes in a will not produce oscillatory motion.

Example. Page 208 Number 1. An 8-lb weight is attached to the lower end of a coil spring suspended from the ceiling and comes to rest in its equilibrium position, thereby stretching the spring 0.4 ft. The weight is then pulled down 6 in. below its equilibrium position and released at $t = 0$. The resistance of the medium in pounds is numerically equal to $2x'$, where x' is the instantaneous velocity in feet per second.

(a) Set up the differential equation for the motion and list the initial conditions.

Solution. For the spring constant, we have $F = kx$, so

$$8 \text{ lb} = x(0.4 \text{ ft}), \text{ or } k = 20 \text{ lb/ft}.$$

The mass satisfies $m = $ force/acceleration, so

$$m = 8 \text{ lb}/32 \text{ ft/sec}^2 = 1/4 \text{ lb sec}^2/\text{ft}.$$

Notice $a = 2$ lb sec/ft, $x(0) = 1/2$ ft, and $x'(0) = 0$ ft/sec^2. So the DE is

$$x'' + \underbrace{2b}_{a/m} x' + \underbrace{\lambda^2}_{k/m} = 0 \text{ or } x'' + \frac{2}{1/4}x' + \frac{20}{1/4}x = 0,$$

or, simplifying,

$$x'' + 8x' + 80x = 0.$$

(b) Solve the initial-value problem set up in part (a) to determine the displacement of the weight as a function of the time.

Solution. The auxiliary equation is $m^2+8n+80 = 0$ (here m is a dummy variable, not mass). We find

$$m = \frac{-(8) \pm \sqrt{(8)^2 - 4(1)(80)}}{2(1)} = \frac{-8 \pm \sqrt{256}}{2} = -4 \pm 8i.$$

The general solution of the DE is

$$x = e^{-4t}(c_1 \sin 8t + c_2 \cos 8t).$$

Applying initial conditions $x(0) = 1/2$ ft and $x'(0) = 0$ ft/sec^2, we find:

$$x = e^{-4t}\left(\frac{1}{4}\sin 8t + \frac{1}{2}\cos 8t\right) = \frac{\sqrt{5}}{4}e^{-4t}\cos(8t - \varphi) \text{ where } \varphi = \cos^{-1}(2/\sqrt{5}).$$

Revised: 3/4/2019