Section 5.4. Forced Motion

Note. In this section we consider a special case of forced motion where the external force satisfies: $F(t) = F_1 \cos \omega t$.

Note. We can rewrite the DE as

$$mx'' + ax' + kx = F_1 \cos \omega t.$$

So

$$x'' + 2bx' + \lambda^2 x = E_1 \cos \omega t,$$

where $2b = 1/m$, $\lambda^2 = k/m$, and $E_1 = F_1/m$. The complimentary function is (from Section 5.3)

$$x_c = ce^{-bt} \cos\left(\sqrt{\lambda^2 - b^2}t + \varphi\right).$$

By the method of underdetermined coefficients we get

$$x_p = \frac{E_1}{\sqrt{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}} \cos(\omega t - \theta)$$

where

$$\cos \theta = \frac{\lambda^2 - \omega^2}{\sqrt{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}} \text{ and } \sin \theta = \frac{2b\omega}{\sqrt{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}}.$$

So

$$x = ce^{-bt} \cos\left(\sqrt{\lambda^2 - b^2}t + \varphi\right) + \frac{E_1}{\sqrt{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}} \cos(\omega t - \theta).$$

Notice that the first term rapidly approaches 0 as t increases. For this reason, the second term is called the *steady state term*.