Section 5.6. Electric Circuit Problems

Note. In this section, we consider certain second order lienar DEs which result from an elementary circuit and Kirchoff's Voltage Law.

Note. Recall:

1. *Electromotive force* (emf) is the difference in potential of the two terminals of an energy source (commonly a battery, or cell). We measure emf (or voltage) in volts (V).

2. *Current* is the charge that passes through a cross section of wire in a unit time. Current is measured in amperes and charge is measured in coulombs.

3. The *resistance* to the flow of current is measured in ohms (Ω).

4. An *inductor* coil prevents a battery from instantaneously establishing a current through it. The battery (emf) has to do work against the coil as the current flows through it. Inductance is measured in henrys (H).

5. A *capacitor* consists of two pieces of metal (usually plates) that store charge. Capacitance is measured in farads (or coulombs per volt).

Definition/Note. The voltage drops across a resistor is $E_R = Ri$ where R is the resistance and i is current. The voltage drop across an inductor is $E_L = Li'$ where L is the inductance (and the prime indicates a derivative with respect to time). The voltage drop across a capacitor is $E_C = \frac{1}{C}q$ where C is the capacitance and q is the instantaneous charge on the capacitor. From the definitions above, $q' = i$ and so $E_C = \frac{1}{C}\int i\,dt$.

Note. *Kirchoff's Voltage Law* relates these voltage drops: The sum of the voltage drops across resistors, inductors, and capacitors is equal to the total emf in a closed circuit.

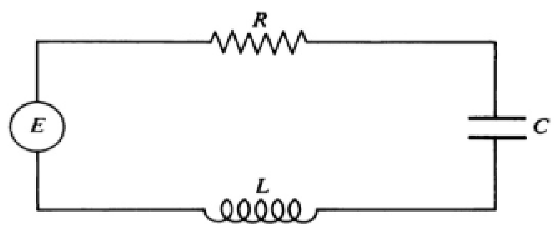

FIGURE 5.12

This circuit, has the equation (from Kirchoff's Law)

$$Li' + Ri + \frac{1}{C}q = E.$$

Both i and q are functions of time, so we can write (since $i = q'$):

$$Lq'' + Rq' + \frac{1}{C}q = E$$

where q is charge at time t.

Note. We can also write a DE in terms of current i: $Li' + Ri + \dfrac{1}{C}q = E$ or

$$Li'' + Ri' + \frac{1}{C}q' = E',$$

or

$$Li'' + Ri' + \frac{1}{C}i = E'.$$

Notice that we have two second order linear DEs, one involving charge q and one involving current i If the current involves no capacitor, then we get the DE governing the so called LR-circuit:

$$Li' + RiE.$$

Also, if there is no inductor, we have

$$Rq' + \frac{1}{C}q = E.$$

Example. Page 232 Number 5. A circuit has in series of electromotive force given by $E(t) = 100\sin 200t$ V, a resistor of 40 Ω, an inductor of 0.25 H, and a capacitor of 4×10^{-4} farads. If the initial current is zero, and the initial charge on the capacitor is 0.01 coulombs, find the current at any time $t > 0$.

Solution. symbolically, we have $E(t) = 100\sin 200t$, $R = 40$, $L = 0.25$, $C = 4 \times 10^{-4}$, $q(0) = 0.01$, and $i(0) = q'(0) = 0$. So using

$$Lq'' + Rq' + \frac{1}{C}q = E$$

we get $0.25q'' + 40q' + 2500a = 100\sin 200t$ or

$$q'' + 160q' + 10,000q = 400\sin 200t.$$

The associated homogeneous DE is $q'' + 160q' + 10,000q = 0$ which has auxiliary equation $m^2 + 160m + 10,000 = 0$ so that we have

$$m = \frac{-160 \pm \sqrt{(160)^2 - 40,000}}{2} = -80 \pm 60i.$$

So

$$q_c = e^{-80t}(c_1 \cos 60t + c_2 \sin 60t).$$

By the method of undetermined coefficients, we look for:

$$\begin{aligned} q_p &= A\sin 200t + B\cos 200t \\ q_p' &= 200A\cos 200t - 200B\sin 200t \\ q_p'' &= -40,000A\sin 200t - 40,000B\cos 200t. \end{aligned}$$

So we need

$$-40,000(A\sin 200t + B\cos 200t) + 32,000(A\cos 200t - B\sin 200t)$$

$$+10,000(A\sin 200t + B\cos 200t) = 400\sin 200t$$

or

$$-62,000A\sin 200t + 2,000B\cos 200t = 400\sin 200t.$$

So we must have $A = -1/1555$ and $B = 0$ and $q_p = -(1/155)\sin 200t$. So we have

$$q = e^{-80t}(c_1\cos 60t + c_2\sin 60t) - \frac{1}{155}\sin 200t.$$

Since $q(0) = 0.01$ then $c_1 = 0.01$ and so

$$q = e^{-80t}(0.01\cos 60t + c_2\sin 60t) - \frac{1}{155}\sin 200t.$$

Then

$$q' = -80e^{-80t}(0.01\cos 60t + c_2\sin 60t) + e^{-80t}(-60(0.01)\sin 60t + 60c_2\cos 60t) - \frac{200}{155}\cos 200t$$

$$= e^{-80t}(-0.8\cos 60t - 80c_2\sin 60t - 0.6\sin 60t + 60c_2\cos 60t) - \frac{40}{31}\cos 200t$$

$$= e^{-80t}((-0.8 + 60c_2)\cos 60t + (-80c_2 - 0.6)\sin 60t) - \frac{40}{31}\cos 200t.$$

So $q'(0) = -0.8 + 60c_2 - 40/31 = 0$ or $60c_2 = 8/10 + 40/31 = 648/310 = 324/155$ and $c_2 = 324/9300 = 27/775$. So

$$q = e^{-80t}\left(0.1\cos 60t + \frac{27}{775}\sin 60t\right) - \frac{1}{155}\sin 200t$$

and, since $-0.8 + 60c_2 = 40/31$ and $-80c_2 - 0.6 = -105/31$,

$$i = q' = e^{-80t}((-0.8 + 60c_2)\cos 60t + (-80c_2 - 0.6)\sin 60t) - \frac{40}{31}\cos 200t$$

$$= e^{-80t}\left(\frac{40}{31}\cos 60t - \frac{105}{31}\sin 60t\right) - \frac{40}{31}\cos 200t.$$

Notice that as time increases, the first term in i rapidly approaches zero (this is called the *transient term*) and the second term (the *steady state term*) dominates.

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