Chapter 6. Series Solutions of Linear Differential Equations

Section 6.1. Power Series Solutions about an Ordinary Point

Note. In this section we search for solutions of the second order linear homogeneous DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

which can be expressed in the form

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - x_0)^n.$$

This is called a *power series* in $(x - x_0)$.

Note. We need to explore when a DE has a solution of the above form. We simplify things a bit. The above DE can be rewritten as

$$y'' + P_1(x)y' + P_2(x)y = 0$$

where $P_1(x) = a_1(x)/a_0(x)$ and $P_2(x) = a_2(x)/a_0(x)$.

Definition. A function f is said to be *analytic at* x_0 if its Taylor series about x_0, $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ exists and converges to $f(x)$ for all x in some open interval containing x_0.

Definition. The point x_0 is an *ordinary point* of the DE $y'' + P_1(x)y' + P_2(x)y = 0$ if P_1 and P_2 are analytic at x_0. If either of these functions is not analytic at x_0 then x_0 is a *singular point* of the DE.

Note. A polynomial function is analytic everywhere. So are the functions e^x, $\sin x$, and $\cos x$. A quotient of polynomials is analytic except for points where the denominator is 0.

Theorem 6.1. If x_0 is an ordinary point of the above DE then the DE has two nontrivial linearly independent solutions of the form $\sum_{n=1}^{\infty} c_n(x - x_0)^n$ and this power series converge is some interval $|x - x_0| < R$ where $R > 0$.

Note. The method we use is rather straightforward, although computationally long. WE simple assume a power series solution and *force* it to be a solution.

Example. Page 249 Number 10. Find a power series solution of

$$y'' - (x^2 + x)y' + y = 0.$$

Solution. Notice that $P_1(x)$ and $P_2(x)$ are analytic everywhere so every point x_0 is an ordinary point. Lets take $x_0 = 0$. Let $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Plugging into the DE gives:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (x^2 + x)\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^{n+1} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or, reindexing,

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)c_{n-1}x^n - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or, collecting together powers of x,

$$\begin{aligned}(2c_2 + (3)(2)c_3 x) - (c_1 x) + (c_0 + c_1 x)\\ + \sum_{n=2}^{\infty}\left((n+2)(n+1)c_{n+2} - (n-1)c_{n-1} - nc_n + c_n\right)x^n = 0\end{aligned}$$

or

$$(2c_2 + c_0) + (6c_3)x + \sum_{n=2}^{\infty}\left((n+2)(n+1)c_{n+2} - (n-1)c_{n-1} - nc_n + c_n\right)x^n = 0.$$

This is true for all x, so we need:

$$\begin{aligned} 2c_2 + c_0 &= 0 \\ 6c_3 &= 0 \\ c_{n+2} &= \frac{(n-1)(c_{n-1}+c_n)}{(n+2)(n+1)} \text{ for } n \geq 2 \\ &\text{or} \\ c_0 &= c_0 \\ c_1 &= c_1 \\ c_3 &= 0 \\ c_4 &= \frac{c_1 + c_2}{(4)(3)} = \frac{c_1 - c_0/2}{12} = \frac{c_1}{12} - \frac{c_0}{24} \\ c_5 &= \frac{2(c_2 + c_3)}{(5)(4)} = \frac{1c_0/2 + 0}{10} = -\frac{1}{20}c_0 \\ c_6 &= \frac{-c_0}{240} + \frac{c_1}{120} \\ &\vdots \end{aligned}$$

So

$$y = c_0 + c_1 x + \left(\frac{-1}{2}c_0\right)x^2 + \left(\frac{-c_0}{24} + \frac{c_1}{12}\right)x^4 + \left(\frac{-1}{20}c_0\right)x^5 + \left(\frac{-c_0}{240} + \frac{c_1}{120}\right)x^6 + \cdots$$
$$= c_0\left(a - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{240}x^6 + \cdots\right) + c_1\left(x + \frac{1}{12}x^4 + \frac{1}{120}c^6 + \cdots\right).$$

(We have rearranged here using the absolute convergence within the radius of convergence of the power series.) Notice that y is a linear combination of two linearly independent solutions, so this is the general solution of the given DE.

Note. "It can be proved" that if $P_1(x)$ converges for $|x - x_0| < R)1$ and $P_2(x)$ converges for $|x - x_0| < R_2$ so that the developed series solution to the DE converges for $|x - x_0| < \min\{R_1, R_2\}$. This means that the above series converges for all $x \in \mathbb{R}$ since $P_1(x)$ and $P_2(x)$ had power series which converged for all x (both were polynomials in the previous example).

Example. Page 249 Number 24. The DE $(1 - x^2)y'' - 2xy' + m(m+1)y = 0$, where m is a constant, is called *Legendre's differential equation.*

(a) Show that $x = 0$ is an ordinary point of this DE, and find two linearly independent power series solutions in powers of x.

Solution. Notice that $P_1(x) = -2/(1 - x^2)$ and $P_2(x) = m(m+1)/(1 - x^2)$, so $P_1(x)$ and $P_2(x)$ are analytic at $x = 0$ and so $x = 0$ is an ordinary point of the DE. Suppose $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$.

Plugging into the DE gives:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + m(m+1)\sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=1}^{\infty} nc_n x^n + m(m+1)\sum_{n=0}^{\infty} c_n x^n = 0$$

or, reindexing,

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=1}^{\infty} nc_n x^n + m(m+1)\sum_{n=0}^{\infty} c_n x^n = 0$$

or, collecting together powers of x,

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + m(m+1)(c_0+c_1x) - 2c_1x - \sum_{n=2}^{\infty}(n(n-1)+2n-m(m+1))c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + m(m+1)c_0 + (m(m+1)c_1 - 2c_1)\,x$$
$$-\sum_{n=2}^{\infty}(n^2+n-m(m+1))c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty}(n^2+n-m(m+1))c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty}(n-m)(n+m+1)c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)c_{n+2} - (n-m)(n+m+1)c_n\right)x^n = 0.$$

This is true for all x, so we need:

$$\begin{aligned} c_0 &= c_0 \\ c_1 &= c_1 \\ c_{n+2} &= \frac{c_n(n-m)(n+m+1)}{(n+2)(n+1)} \text{ for } n \geq 0. \end{aligned}$$

With these recursive formulae, we find the solutions:

$$y_1(x) = c_0\left(1 - \frac{m(m+1)}{2!}x^2 + \frac{(m-2)m(m+1)(m+2)}{4!}x^4 - \cdots\right)$$

$$y_2(x) = c_1\left(x - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!}x^5 - \cdots\right).$$

(b) Show that if m is a nonnegative integer, then one of the two solutions found in part (a) is a polynomial of degree m.

Solution. If m is a nonnegative integer, then either $y_1(x)$ or $y_2(x)$ is an m degree polynomial called the mth degree Legendre polynomial, $P_m(x)$. We find, with the condition $P(1) = 1$, that

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ &\vdots \end{aligned}$$