Section 6.2. Solutions About Singular Points; The Method of Frobenius

Note. We again consider the DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

This time we wish to find a solution defined "near" a singular point x_0 where $a_0(x_0) = 0$. For this, we give a classification of singular points.

Definition. With the notation established, let x_0 be a singular point of the above DE. If the functions

$$\begin{cases} \lim_{x \to x_0}(x - x_0)P_1(x) & \text{if } x = x_0 \\ (x - x_0)P_1(x) & \text{if } x \neq x_0 \end{cases}$$

and

$$\begin{cases} \lim_{x \to x_0}(x - x_0)^2 P_2(x) & \text{if } x = x_0 \\ (x - x_0)^2 P_2(x) & \text{if } x \neq x_0 \end{cases}$$

are both analytic at x_0, then x_0 is a *regular singular point* of the DE. If either of these new functions is not analytic at x_0, then x_0 is an *irregular singular point* of the DE.

Note. We follow the notation of Ross and denote the "new" functions as $(x - x_0)P_1(x)$ and $(x - x_0)P_2(x)$, even though they are defined as

$$(x - x_0)P_1(x) = \begin{cases} \lim_{x \to x_0}(x - x_0)P_1(x) & \text{if } x = x_0 \\ (x - x_0)P_1(x) & \text{if } x \neq x_0 \end{cases}$$

and

$$(x-x_0)^2P_2(x) = \begin{cases} \lim_{x\to x_0}(x-x_0)^2P_2(x) & \text{if } x = x_0 \\ (x-x_0)^2P_2(x) & \text{if } x \neq x_0 \end{cases}$$

Note. We can find series solutions about such singular points.

Theorem 6.2. Suppose x_0 is a regular singular point of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Then the DE has at has at least one nontrivial solution of the form

$$|x-x_0|^R \sum_{n=0}^{\infty} c_n(x-x_0)^n$$

where R is a (real or complex) constant which may be determined. This solution is valid in some deleted interval $0 < |x-x_0| < s$ where $s > 0$.

Note. We use the Method of Frobenius when we apply Theorem 6.2.

Example. Page 254 Number 11. Find solutions of

$$2x^2y'' - xy' + (x-5)y = 0$$

in some deleted interval $0 < x < R$.

Solution. Notice that $P_1(x) = -x/(2x^2)$ and $P_2(x) = (x-5)/(2x^2)$. Now with $x_0 = 0$, we have

$$xP_1(x) \;=\; \frac{-x^2}{2x^2} = \frac{-1}{2}$$

$$x^2P_2(x) = \frac{x^2(x-5)}{2x^2} = \frac{x-5}{2}$$

where we use the red $=$ as described above. So both of these new functions are analytic at $x = 0$. That is, $x = 0$ is a regular singular point. So assume

$$y = |x|^R \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+R}$$

(ignoring the absolute value for now) is a solution. Then

$$y' = \sum_{n=0}^{\infty}(n+R)c_n x^{n+R-1} \text{ and } y'' = \sum_{n=0}^{\infty}(n+R)(n+R-1)c_n x^{n+R-2}$$

Plugging these series into the DE gives

$$2\sum_{n=0}^{\infty}(n+R)(n+R-1)c_n x^{n+R} - \sum_{n=0}^{\infty}(n+R)c_n x^{n+R} + \sum_{n=0}^{\infty} c_n x^{n+R-1} - 5\sum_{n=0}^{\infty} c_n x^{n+R} = 0.$$

Simplifying we get:

$$(2R(R-1)-R-5)c_0x^R + \sum_{n=1}^{\infty}\left((2(n+R)(n+R-1)-(n+R)-5)c_n + c_{n-1}\right)x^{n+R} = 0.$$

This means that

$$2R(R-1) - R - 5 = 2R^2 - 3R - 5 = (2R-5)(R+1) = 0 \text{ or } R = -1, 5/2.$$

This is called the *indicial equation*. So we get the recurrence formula

$$c_n = \frac{-c_{n-1}}{2(n+R)(n+R-1)-(n+R)-5} \text{ for } n \geq 1.$$

With $R = 5/2$, the recurrence formula becomes:

$$c_n = \frac{-c_{n-1}}{n(2n+7)} \text{ for } n \geq 1.$$

Notice that c_0 is arbitrary, and we get

$$y = c_0\left(x^{5/2} - \frac{1}{9}x^{7/2} + \frac{1}{198}x^{9/2} - \frac{1}{7722}x^{11/2} + \cdots\right).$$

If we use $R = -1$, we get

$$y = c_0 \left(x^{-1} + \frac{1}{5} + \frac{1}{30}x + \frac{1}{90}x^2 + \cdots \right).$$

Again c_0 is arbitrary. So the general solution is:

$$y = k_1 \left(x^{5/2} - \frac{1}{9}x^{7/2} + \frac{1}{198}x^{9/2} - \frac{1}{7722}x^{11/2} + \cdots \right) + k_2 \left(x^{-1} + \frac{1}{5} + \frac{1}{30}x + \frac{1}{90}x^2 + \cdots \right)$$

$$= k_1 x^{5/2} \left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \cdots \right) + k_2 x^{-1} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \cdots \right).$$

Note. Notice that Theorem 6.2 guarantees *at least one* solution of a certain form. This previous example had two solutions of that form. The following theorem clarifies this a bit.

Theorem 6.3. Let x_0 be a regular singular point of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Let R_1 and R_2 be the roots of the indicial equation (where $R_1 > R_2$ for R_1 and R_2 real). Then

1. If $R_1 - R_2 \notin \{0, 1, 2, \ldots\}$ then the DE has two nontrivial independent solutions y_1 and y_2 of the forms:

$$y_1(x) = |x - x_0|^{R_1} \sum_{n=0}^{\infty} c_n(x - x_0) \text{ and } y_2(x) = |x - x_0|^{R_2} \sum_{n=0}^{\infty} c_n^*(x - x_0)^n$$

where $c_0 \neq 0$ and $c_0^* \neq 0$.

2. If $R_1 - R_2 \in \{1, 2, 3, \ldots\}$ then the DE has two nontrivial linearly independent solutions y_1 and y_2 of the forms:

$$y_1(x) = |x - x_0|^{R_1} \sum_{n=0}^{\infty} c_n(x - x_0) \text{ and}$$

$$y_2(x) = |x - x_0|^{R_2} \sum_{n=0}^{\infty} c_n^*(x - x_0)^n + cy_1(x) \ln |x - x_0|$$

where $c_0 \neq 0$, $c_0^* = 0$, an dc is a constant (possibly 0).

3. If $R_1 = R_2$ then the DE has two nontrivial linearly independent solutions y_1 and y_2 of the forms

$$y_1(x) = |x - x_0|^{R_1} \sum_{n=0}^{\infty} c_n(x - x_0) \text{ and}$$

$$y_2(x) = |x - x_0|^{R_1+1} \sum_{n=0}^{\infty} c_n^*(x - x_0)^n + y_1(x) \ln |x - x_0|$$

where $c_0 \neq 0$.

In each of these cases, the solutions are valid for $0 < |x - x_0| < R$ for some $R > 0$.

Example. Page 270 Number 16. Find solutions of

$$x^2y'' + xy' + (x^2 - 1/4)y = 0$$

in some deleted interval $0 < x < R$.

Solution. Notice that $x_0 = 0$ is a regular singular point. So suppose $y = \sum_{n=0}^{\infty} c_n x^{n+R}$. Then calculating derivatives and plugging them into the DE gives

$$\left(R(R-1) + R - \frac{1}{4}\right) c_0 x^R + \left((1+R)^2 - \frac{1}{4}\right) c_1 x^{1+R}$$

$$+\sum_{n=2}^{\infty}\left(\left((n+R)(n+R-1)+(n+R)-\frac{1}{4}\right)c_n+c_{n-2}\right)x^{n+R}=0.$$

So from the indicial equation:

$$R(R-1)+R-\frac{1}{4}=0 \text{ and } R_1=\frac{1}{2}, R_2=\frac{-1}{2}.$$

Notice that $R_1 - R_2 = 1$ and Case 2 of Theorem 6.3 applies. Using $R = 1/2$, we find that (as in the previous example):

$$y_1(x)=c_1x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^5}{120}-\cdots\right).$$

Now, we *hope* that for $R=-1/2$, we get $y_2(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$. If this leads to a contradiction, we will have to use reduction of order using $y_1(x)$. With $R=-1/2$ we have $c_0=c_0$, $c_1=c_1$ (that is, c_0 and c_1 are arbitrary), and

$$c_n=\frac{-c_{n-2}}{n^2-n} \text{ for } n\geq 2.$$

We get

$$y_2(x)=c_0x^{-1/2}\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)+c_1x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^5}{120}-\cdots\right).$$

Notice that the second part of $y_2(x)$ is $y_1(x)$. In fact, $y_2(x)$ *is* the general solution of the given DE.

Note. The previous example illustrates the fact that when $R_1 - R_2$ is a positive integer, it may be the case that the smaller root R_2 may generate the general solution. Therefore, it is a good habit to always use the smaller root first.

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