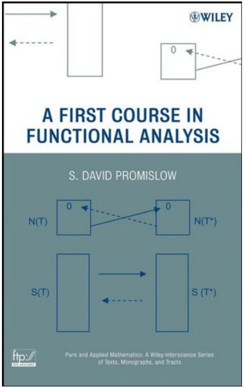
Introduction to Functional Analysis

Chapter 1. Linear Spaces and Operators

1.3. Linear Operators—Proofs of Theorems



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Proof. Suppose $N(T) = 0$ and $Tx = Ty$. Then $0 = Tx - Ty = T(x - y)$ since T is linear. So $x - y \in N(T)$ and so $x - y = 0$. Hence $x = y$. That is, T is injective as claimed.

Suppose T is injective (that is, $Tx = Ty$ implies $x = y$). We know that $T(0) = 0$, so if $z \in N(T)$ then $Tz = T(0) = 0$ and so $z = 0$. Hence $N(T) = 0$, as claimed. $\square$

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Proof. Let $z_1, z_2 \in R(T)$. Say $z_1 = Tx_1$ and $z_2 = Tx_2$. Then for $\alpha, \beta \in \mathbb{F}$, $\alpha z_1 = \alpha T(x_1) = T(\alpha x_1)$ and $\beta z_2 = \beta T(x_2) = T(\beta x_2)$ since T is linear. So

$$\alpha z_1 + \beta z_2 = T(\alpha x_1) + T(\beta x_2) = T(\alpha x_1 + \beta x_2)$$

and hence

$$T^{-1}(\alpha z_1 + \beta z_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}(z_1) + \beta T^{-1}(z_2).$$

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