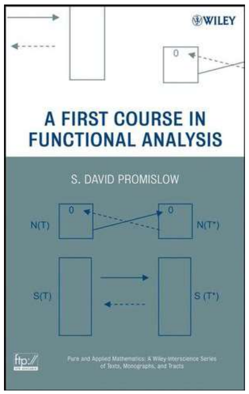
# Introduction to Functional Analysis

**Chapter 2. Normed Linear Spaces: The Basics**

2.12. Fixed Points and Contraction Mappings—Proofs of Theorems








## Proposition 2.44

**Proposition 2.44. The Contraction Mapping Theorem.** A contraction mapping $T$ from a complete metric space to itself has a unique fixed point.

**Proof.** Let $x \in X$. Inductively we have

$$d(T^{n+1}x, T^n x) \le cd(T^n x, T^{n-1}x)$$
$$\le c^2 d(T^{n-1}x, T^{n-2}x) \le \cdots \le c^n d(Tx, x).$$

Since $c < 1$, the sequence $(T^n x)_{n=1}^{\infty}$ is fast Cauchy and hence Cauchy by Proposition 2.10(a). Since the space is complete, there is $y \in X$ such that $(T^n x) \to y$. Now a contraction is continuous (just take $\delta = \varepsilon$ in the definition of continuity) and so

$$\lim_{n\to\infty} T(T^n x) = T\left(\lim_{n\to\infty} T^n x\right) = Ty.$$

Also, the sequence $T^{n+1}(x) \to y$ and so we have $Ty = y$ and so $y$ is a fixed point.





## Proposition 2.44

**Proposition 2.44 (continued). The Contraction Mapping Theorem.** A contraction mapping $T$ from a complete metric space to itself has a unique fixed point.

**Proof (continued).** Next, suppose $T$ has two fixed points, say $y$ and $z$. Then

$$d(y,z) = d(Ty, Tz) \le cd(y,z).$$

Since $c < 1$, this can only hold if $d(y,z) = 0$ and $y = z$. So the fixed point is unique. □





## Corollary 2.45

**Corollary 2.45.** If $T$ is a mapping from a complete metric space to itself such that $T^n$ is a contraction for some $n \in \mathbb{N}$, then $T$ has a unique fixed point.

**Proof.** Let $y$ be the unique fixed point of $T^n$, so that $T^n y = y$. Then $T^n(Ty) = T(T^n y) = Ty$, and so $Ty$ is also fixed by $T^n$. But the uniqueness hypothesis then implies that $y = Ty$ so that $y$ is also a fixed point of $T$.

Notice that if $z$ is a fixed point of $T$ so that $Tz = z$, then $T^n z = z$ and $z$ is also a fixed point of $T^n$. Again, the uniqueness hypothesis implies that $z = y$. So the fixed point of $T$ is unique, as claimed. □