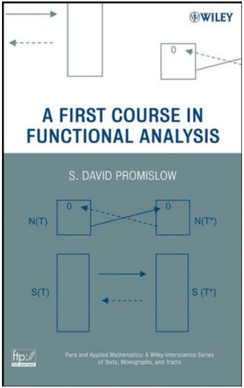
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.2. Norms—Proofs of Theorems



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Theorem 2.3. Continuity of Operations

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Suppose that (x_n) and (y_n) are sequences in a normed linear space, and (α_n) is a sequence in $\mathbb{F}$, and that $x = \lim(x_n)$, $y = \lim(y_n)$, and $\alpha = \lim(\alpha_n)$. Then

(a) $\lim(x_n + y_n) = \lim(x_n) + \lim(y_n) = x + y$.

(b) $\lim(\alpha_n x_n) = \lim(\alpha_n) \lim(x_n) = \alpha x$.

(c) $\lim \|x_n\| = \|x\|$.

The proofs of (a) and (c) are to be given in Exercise 2.2.

Theorem 2.3 (continued)

Proof. (b) Since (α_n) is convergent, it is bounded (a standard result from senior level Analysis 1; see Theorem 2.3 in my online notes for Analysis 1 on Section 2.1. Sequences of Real Numbers) and there is $K \in \mathbb{R}^+$ such that $|\alpha_n| \le K$ for all n.

By hypothesis, $\lim \|x_n - x\| = 0$ and $\lim |\alpha_n - \alpha| = 0$. So

$$\begin{aligned}
\lim \|\alpha_n x_n - \alpha x\| &= \lim \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\
&\le \lim \|\alpha_n x_n - \alpha_n x\| + \lim \|\alpha_n x - \alpha x\| \\
&\quad \text{by the Triangle Inequality} \\
&= \lim |\alpha_n| \|x_n - x\| + \lim \|x\| |\alpha_n - \alpha| \\
&\quad \text{by the Scalar Property} \\
&\le K \lim \|x_n - x\| + \|x\| \lim |\alpha_n - \alpha| = 0.
\end{aligned}$$

So $\lim(\alpha_n x_n) = \alpha x$. □

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Proposition 2.5

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(i) $B(x;r)$ is open where $r > 0$.

(ii) The closure of $B(x;r)$ is $\overline{B}(x;r)$.

Proof. **(i)** Let $y \in B(x;r)$. Then $\|y - x\| < r$, and so there is positive s with $s < r - \|x - y\|$:

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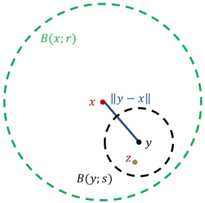

If $\|z-y\| < s$ then

$$\|z-x\| = \|z-y+y-x\| \leq \|z-y\| + \|y-x\| < s + (r-s) = r.$$

So $B(y;s) \subseteq B(x;r)$ and y is an interior point of $B(x;r)$. Therefore $B(x;r)$ is open. $\square$

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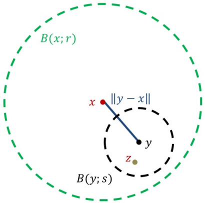

If $\|z - y\| < s$ then

$$\|z - x\| = \|z - y + y - x\| \leq \|z - y\| + \|y - x\| < s + (r - s) = r.$$

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Proof (continued). (ii) Let (z_n) be a sequence in $\overline{B}(x;r)$ that converges to z (so z is a limit point of $\overline{B}(x;r)$). Then the sequence $(z_n - x)$ converges to $z - x$. By continuity of the norm (Theorem 2.3(c)), we have $\|z - x\| \leq r$ (since $\|z_n - x\| \leq r$ for all $n \in \mathbb{N}$); that is, $z \in \overline{B}(x;r)$ and $\overline{B}(x;r)$ contains its limit points. So by Note 2.2.B, $\overline{B}(x;r)$ is closed. Hence, by the definition of closure of a set, $\overline{B}(x;r)$ contains the closure of $B(x;r)$.

Proposition 2.5 (continued 2)

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(i) $B(x;r)$ is open where $r > 0$.

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Proof (continued). Conversely, if $\|y - x\| = r$ then define the sequence $y_n = x + (1 - \frac{1}{n})(y - x)$. Then

$$\|y_n - x\| = \|(1 - 1/n)(y - x)\| = |1 - 1/n|\|y - x\| < \|y - x\| = r,$$

and so $y_n \in B(x;r)$. Moreover, by the continuity of addition and scalar multiplication (Theorem 2.3(a) and (b)) we have

$$\lim(y_n) = \lim(x + (1 - 1/n)(y - x)) = x + (y - x) = y.$$

So by Theorem 2.2.A(iii), y is in the closure of $B(x;r)$ and hence the closure of $B(x;r)$ contains $\overline{B}(x;r)$. Therefore the closure of $B(x;r)$ is $\overline{B}(x;r)$, as claimed. $\square$

Theorem 2.2.B. The Compact Set Theorem

Theorem 2.2.B. The Compact Set Theorem If $K \subseteq X$, X a normed linear space, is compact then K is closed and bounded.

Proof. Suppose K is not bounded. Fix $a \in X$. Then for each $n \in \mathbb{N}$, $B(a; n)^c$ contains some $k_n \in K$. Then sequence (k_n) diverges "to infinity" (recall that convergent sequences are bounded), and so by (ii) of the definition of "compact," K is not compact, a contradiction. So the assumption that K is not bounded is false, and hence K is bounded.

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Next, let $x \in \overline{K}$ and suppose K is compact. Then by Theorem 2.2.A(iii), there is a sequence $(x_n) \subseteq K$ such that $(x_n) \to x$. By the definition of "compact" part (ii) there is a subsequence (x_{n_k}) of (x_n) which converges to a point in K. But (x_{n_k}) must converge to x since a subsequence of a convergent sequence has the same limit as the sequence itself. Therefore $x \in K$. So $K = \overline{K}$ and K is closed. □

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