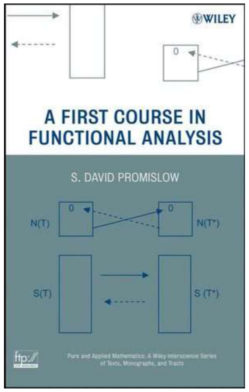
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.3. Space of Bounded Functions—Proofs of Theorems



Theorem 2.3.A

Theorem 2.3.A. $B(S)$ is a normed linear space.

Proof. Let $f, g \in B(S)$. Then for $s \in S$,

$$\begin{aligned} |(f+g)(s)| &= |f(s)+g(s)| \\ &\leq |f(s)|+|g(s)| \text{ by the Triangle Inequality on } \mathbb{R} \\ &\leq \|f\|+\|g\| < \infty. \end{aligned}$$

So $\|f+g\| \leq \|f\|+\|g\|$ and $f+g \in B(S)$. For $\alpha \in \mathbb{R}$ and $f \in B(S)$, $|\alpha f(s)| = |\alpha||f(s)|$, so

$$\|\alpha f\| = \sup\{|\alpha f(s)| \mid s \in S\} = \sup\{|\alpha||f(s)| \mid s \in S\}$$

$$= |\alpha| \sup\{|f(s)| \mid s \in S\} = |\alpha|\|f\| < \infty,$$

and $\alpha f \in B(S)$. Therefore, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in B(S)$, we have $\alpha f + \beta g \in B(S)$ and so $B(S)$ is a linear space.

Theorem 2.3.A (continued)

Theorem 2.3.A. $B(S)$ is a normed linear space.

Proof (continued). To show that $\|\cdot\|$ is a norm, the Triangle Inequality is established above. Also, $\|\alpha f\| = |\alpha|\|f\|$ is above. Finally, if $\|f\| = 0$ then $\sup\{|f(s)| \mid s \in S\} = 0$ and so $f \equiv 0$. So $\|\cdot\|$ is a norm, and $B(S)$ is a normed linear space under $\|\cdot\|$. □