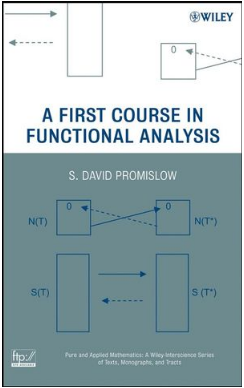
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.3. Space of Bounded Functions—Proofs of Theorems



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Theorem 2.3.A

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Proof. Let $f, g \in B(S)$. Then for $s \in S$,

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and $\alpha f \in B(S)$. Therefore, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in B(S)$, we have $\alpha f + \beta g \in B(S)$ and so $B(S)$ is a linear space.

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Theorem 2.3.A (continued)

Theorem 2.3.A. $B(S)$ is a normed linear space.

Proof (continued). To show that $\| \cdot \|$ is a norm, the Triangle Inequality is established above. Also, $\|\alpha f\| = |\alpha|\|f\|$ is above. Finally, if $\|f\| = 0$ then $\sup\{|f(s)| \mid s \in S\} = 0$ and so $f \equiv 0$. So $\| \cdot \|$ is a norm, and $B(S)$ is a normed linear space under $\| \cdot \|$. □