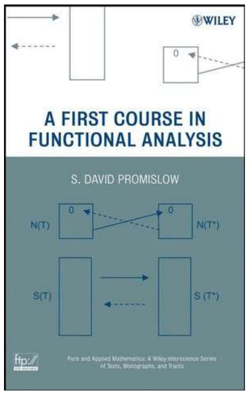
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.4. Bounded Linear Operators—Proofs of Theorems



Theorem 2.6

Theorem 2.6. Given $T \in \mathcal{L}(X, Y)$, the following are equivalent:

(i) T is uniformly continuous on X;

(ii) T is continuous at some point $x \in X$;

(iii) T is bounded.

Proof. The proof of **(i)** $\Rightarrow$ **(ii)** is trivial.

(ii) $\Rightarrow$ **(iii).** Suppose T is continuous at some point x. With $\varepsilon = 1$, there is $\delta > 0$ such that $T(B(x; 2\delta)) \subseteq B(T(x); 1)$. Let $z \in X$ be a unit vector, $\|z\| = 1$. Then $x + \delta z \in B(x, 2\delta)$, and so

$$T(x + \delta z) = T(x) + \delta T(z) \in B(T(x); 1).$$

Hence $\delta \|Tz\| = \|T(x + \delta z) - T(x)\| < 1$ and $\|Tz\| < 1/\delta$. Since z with $\|z\| = 1$ was arbitrary, T is bounded and $\|T\| \leq 1/\delta$.

Theorem 2.6 (continued)

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(i) T is uniformly continuous on X;

(ii) T is continuous at some point $x \in X$;

(iii) T is bounded.

Proof (continued). (iii) $\Rightarrow$ **(i).** Suppose T is bounded, say $\|T\| = K$. Let $\varepsilon > 0$ and take $\delta = \varepsilon / K$. Then for any $x, y \in X$ with $\|y - x\| < \delta$ we have by Note 2.2.A that

$$\|T(x) - T(y)\| \leq \|T\| \|y - x\| < K\delta = K\left(\frac{\varepsilon}{K}\right) = \varepsilon.$$

Therefore, T is uniformly continuous on X. $\square$

Proposition 2.8

Proposition 2.8. For $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ bounded linear operators, $S \circ T = ST$ is linear and $\|ST\| \leq \|S\| \|T\|$.

Proof. For $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{R}$,

$$ST(\alpha x_1 + \beta x_2) = S(T(\alpha x_1 + \beta x_2))$$

$$= S(\alpha T(x_1) + \beta T(x_2)) = \alpha ST(x_1) + \beta ST(x_2),$$

so ST is linear.

For $x \in X$ with $\|x\| = 1$ we have by Note 2.4.A that

$$\|ST(x)\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\| = \|S\| \|T\|.$$

Taking a supremum over all such $x \in X$, the claim follows. $\square$