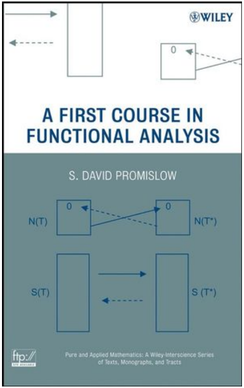
# Introduction to Functional Analysis

**Chapter 2. Normed Linear Spaces: The Basics**

2.4. Bounded Linear Operators—Proofs of Theorems



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# Theorem 2.6

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(ii) $T$ is continuous at some point $x \in X$;

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**(ii)** $\Rightarrow$ **(iii).** Suppose $T$ is continuous at some point $x$. With $\varepsilon = 1$, there is $\delta > 0$ such that $T(B(x; 2\delta)) \subseteq B(T(x); 1)$. Let $z \in X$ be a unit vector, $\|z\| = 1$. Then $x + \delta z \in B(x, 2\delta)$, and so

$$T(x + \delta z) = T(x) + \delta T(z) \in B(T(x); 1).$$

Hence $\delta\|Tz\| = \|T(x + \delta z) - T(x)\| < 1$ and $\|Tz\| < 1/\delta$. Since $z$ with $\|z\| = 1$ was arbitrary, $T$ is bounded and $\|T\| \leq 1/\delta$.

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**Proof (continued)**. **(iii) $\Rightarrow$ (i).** Suppose $T$ is bounded, say $\|T\| = K$. Let $\varepsilon > 0$ and take $\delta = \varepsilon/K$. Then for any $x, y \in X$ with $\|y - x\| < \delta$ we have by Note 2.2.A that

$$\|T(x) - T(y)\| \leq \|T\|\|y - x\| < K\delta = K\left(\frac{\varepsilon}{K}\right) = \varepsilon.$$

Therefore, $T$ is uniformly continuous on $X$. $\square$

# Proposition 2.8

**Proposition 2.8.** For $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ bounded linear operators, $S \circ T = ST$ is linear and $\|ST\| \leq \|S\|\|T\|$.

**Proof.** For $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{R}$,

$$ST(\alpha x_1 + \beta x_2) = S(T(\alpha x_1 + \beta x_2))$$

$$= S(\alpha T(x_1) + \beta T(x_2)) = \alpha ST(x_1) + \beta ST(x_2),$$

so $ST$ is linear.

For $x \in X$ with $\|x\| = 1$ we have by Note 2.4.A that

$$\|ST(x)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\| = \|S\|\|T\|.$$

Taking a supremum over all such $x \in X$, the claim follows. □

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