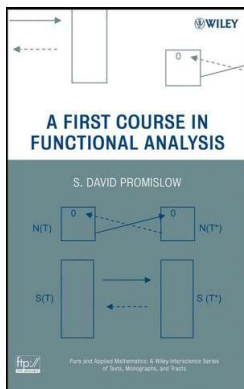


Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.5. Completeness—Proofs of Theorems



Proposition 2.9

Proposition 2.9. In a normed linear space:

- (a) A convergent sequence is Cauchy.
- (b) A Cauchy sequence (x_n) is bounded. That is, there is a $k > 0$ such that $\|x_n\| < k$ for all n .
- (c) All subsequences of a Cauchy sequence are Cauchy.
- (d) If (x_n) is Cauchy and some subsequence converges to x , then (x_n) converges to x .

Proof. (a). Let (x_n) be convergent with limit x . Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|x - x_n\| < \varepsilon/2$. So let $m, n \geq N$. Then

$$\|x_m - x_n\| = \|x_m - x + x - x_n\| \leq \|x_m - x\| + \|x - x_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So (x_n) is Cauchy, as claimed.

Proposition 2.9 (continued 1)

Proposition 2.9(b). In a normed linear space:

- (b) A Cauchy sequence (x_n) is bounded. That is, there is a $k > 0$ such that $\|x_n\| < k$ for all n .

Proof (continued). (b). Suppose (x_n) is Cauchy. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|x_n - x_m\| < \varepsilon = 1$. In particular, with $n = N$ we have $\|x_N - x_m\| < \varepsilon = 1$ and so by the (Backwards) Triangle Inequality $\|x_m\| < \|x_N\| + 1$. Let

$$k = \max\{\|x_1\|, \|x_2\|, \|x_3\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}.$$

Then for all $n \in \mathbb{N}$, $\|x_n\| \leq k$ and so $\{x_n\}$ is bounded, as claimed.

Proposition 2.9 (continued 2)

Proposition 2.9(c). In a normed linear space:

- (c) All subsequences of a Cauchy sequence are Cauchy.

Proof (continued). (c). Let (x_{n_k}) be a subsequence of Cauchy sequence (x_n) . Since (x_n) is Cauchy then, by definition, for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_n - x_m\| < \varepsilon$. For all $k, \ell \geq N$ we have $n_k \geq k \geq N$ (notice that $n_k \geq k$) and $n_\ell \geq \ell \geq N$, so that $\|x_{n_k} - x_{n_\ell}\| < \varepsilon$. Hence $\{x_{n_k}\}$ is Cauchy, as claimed.

Proposition 2.9 (continued 3)

Proposition 2.9(d). In a normed linear space:

- (d) If (x_n) is Cauchy and some subsequence converges to x , then (x_n) converges to x .

Proof (continued). (d). Let (x_n) be Cauchy. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $\|x_n - x_m\| < \varepsilon/2$. Let (x_{n_k}) converge to x . Then there exists $J \in \mathbb{N}$ (where we take, WLOG, $J \geq N$) such that for all $k \geq J$, we have $\|x_{n_k} - x\| \leq \varepsilon/2$ (notice that $n_k \geq k$, so that $n_J \geq J$). In particular, $\|x_{n_J} - x\| \leq \varepsilon/2$. So for $n \geq J \geq N$, we have

$$\|x_n - x\| = \|x_n - x_{n_J} + x_{n_J} - x\| \leq \|x_n - x_{n_J}\| + \|x_{n_J} - x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $(x_n) \rightarrow x$, as claimed. \square

Proposition 2.10

Proposition 2.10.

- (a) A fast Cauchy sequence is Cauchy.
 (b) Any Cauchy sequence contains a fast Cauchy subsequence.

Proof. (a). Let (x_n) be fast Cauchy and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/2^{N-1} < \varepsilon$. Then for $n > m \geq N$ we have

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\| \\ &\leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \leq \sum_{k=m}^{n-1} \frac{1}{2^k} < \sum_{k=m}^{\infty} \frac{1}{2^k} \\ &= \frac{1/2^m}{1 - 1/2} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \varepsilon. \end{aligned}$$

So the sequence is Cauchy, as claimed.

Proposition 2.10 (continued 1)

Proposition 2.10.

- (b) Any Cauchy sequence contains a fast Cauchy subsequence.

Proof (continued). (b) Let (x_n) be Cauchy.

Choose $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$ we have $\|x_m - x_n\| < 1/2$.

Choose N_2 such that $N_2 > N_1$ and for all $m, n \geq N_2$ we have $\|x_m - x_n\| < 1/2^2$.

Now inductively choose $N_k > N_{k-1}$ such that for all $m, n \geq N_k$ we have $\|x_m - x_n\| < 1/2^k$.

Now consider the subsequence (x_{N_k}) . Then for any $k \in \mathbb{N}$ we have $\|x_{N_{k+1}} - x_{N_k}\| \leq 1/2^k$ since $N_{k+1} > N_k \geq N_k$. Therefore (x_{N_k}) is fast Cauchy, as claimed. \square

Proposition 2.11

Proposition 2.11. If $x = \sum_{i=1}^{\infty} x_i$ exists, then $\|x\| \leq \sum_{i=1}^{\infty} \|x_i\|$.

Proof. By continuity of the norm (Theorem 2.3(c)) we have (from the Triangle Inequality)

$$\begin{aligned} \|x\| &= \left\| \lim_{n \rightarrow \infty} s_n \right\| = \lim_{n \rightarrow \infty} \|s_n\| = \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n x_i \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \|x_i\| \right) = \sum_{i=1}^{\infty} \|x_i\|. \end{aligned}$$

Theorem 2.12

Theorem 2.12. A normed linear space X is complete if and only if every absolutely convergent series is convergent.

Proof. Suppose X is complete and (x_i) yields an absolutely convergent series. Then $\sum_{i=1}^{\infty} \|x_i\| < \infty$. Let $\varepsilon > 0$. Now for $N \in \mathbb{N}$ sufficiently large, $\sum_{i=N}^{\infty} \|x_i\| < \varepsilon$. So for $n > m \geq N$ we have for the partial sums

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=m+1}^n x_i \right\| \leq \sum_{i=m+1}^n \|x_i\| \text{ by Triangle Inequality} \\ &\leq \sum_{i=m+1}^{\infty} \|x_i\| < \varepsilon. \end{aligned}$$

So (s_j) is Cauchy and since X is complete, (s_j) is convergent. That is, the absolutely convergent series $\sum x_i$ is convergent. \square

Theorem 2.12 (continued)

Proof (continued). Now suppose that every absolutely convergent series is convergent. Let (x_n) be a fast Cauchy sequence. The series $\sum_{i=1}^{\infty} (x_{i+1} - x_i)$ is absolutely convergent:

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| \leq \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 1,$$

and so is convergent (by hypothesis), say to x . Since the partial sum for this series is $s_n = x_{n+1} - x_1$ and $s_n \rightarrow x$, we have that $x_{n+1} \rightarrow x + x_1$, or $(x_n) \rightarrow x + x_1$. So the fast Cauchy sequence is convergent. Since (x_n) is an arbitrary fast Cauchy sequence, this shows that every fast Cauchy sequence in X is convergent. Let (x'_n) be any Cauchy sequence in X . By Proposition 2.10(b) there is a subsequence (x'_{n_k}) of (x'_n) which is fast Cauchy. As shown above, (x'_{n_k}) converges in X . By Proposition 2.9(d), (x'_n) converges in X . Since (x'_n) is an arbitrary Cauchy sequence in X , then X is complete. \square

Lemma 2.13

Lemma 2.13. Suppose that X is a subspace of the space of all functions from set S to field \mathbb{F} , $F(S)$, and that $\|\cdot\|$ is a norm on X for which the closed unit ball $\overline{B}(1)$ is closed under pointwise limits. That is, if Cauchy sequence $(f_n) \subset \overline{B}(1)$ converges pointwise to f , then $f \in X$ and $f \in \overline{B}(1)$. If a sequence (f_n) in X is Cauchy and converges pointwise to f , then $f \in X$ and (f_n) converges to f with respect to $\|\cdot\|$.

Proof. Let (f_n) be a Cauchy sequence in X which converges pointwise to f . Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|f_n - f_m\| \leq \varepsilon$. Since $\overline{B}(1)$ is closed under pointwise limits, then by scaling $f \in \overline{B}(1)$ to $\varepsilon f / \|f\| \in \overline{B}(\varepsilon)$, we see that $\overline{B}(\varepsilon)$ is closed under pointwise limits. For a given $n_0 \geq N$, the sequence $(f_n - f_{n_0}) \subset \overline{B}(\varepsilon)$ (so this sequence is Cauchy) and converges pointwise to $f - f_{n_0}$ (since (f_n) converges to f pointwise), so by hypothesis $(f - f_{n_0}) \in X$ and $f - f_{n_0} \in \overline{B}(\varepsilon)$. Since X is a subspace, $f = f_{n_0} + (f - f_{n_0}) \in X$ and $\|f - f_{n_0}\| \leq \varepsilon$ for any $n_0 \geq N$. Therefore $(f_n) \rightarrow f$ with respect to $\|\cdot\|$. \square

Theorem 2.14

Theorem 2.14. The space of all bounded functions from set S to field \mathbb{F} (taken to be \mathbb{R} or \mathbb{C}), $B(S)$, is complete with respect to the sup norm.

Proof. Let (f_n) be a Cauchy sequence in $B(S)$. For any point $s \in S$,

$$|f_m(s) - f_n(s)| \leq \sup_{s \in S} |f_m(s) - f_n(s)| = \|f_m - f_n\|.$$

So the sequence $(f_n(s))$ is Cauchy in \mathbb{F} . Since \mathbb{F} is complete, $(f_n(s))$ converges to some point in \mathbb{F} , denoted $f(s)$. So f is the pointwise limit of (f_n) on S . Moreover, if $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$ (or equivalently, $f_n \in \overline{B}(1)$) then for any $s \in S$ we have $|f_n(s)| \leq \|f_n\| \leq 1$, and so $|f(s)| = |\lim_{n \rightarrow \infty} f_n(s)| = \lim_{n \rightarrow \infty} |f_n(s)| \leq 1$. Therefore $f \in \overline{B}(1)$. So by Lemma 2.13, $(f_n) \rightarrow f$ with respect to the sup norm. \square

Theorem 2.16

Theorem 2.16. A subspace Y of a Banach space X is itself a Banach space if and only if Y is closed.

Proof. First, suppose Y is closed and let (y_n) be a Cauchy sequence in Y . Then (y_n) is a Cauchy sequence in X and since X is a Banach space, $(y_n) \rightarrow x$ for some $x \in X$. Since Y is closed then, by Theorem 2.2.A(iii), $x \in Y$ and so (y_n) is convergent in Y and Y is a Banach space.

Conversely, suppose Y is not closed. Then there is $x \in \bar{Y}$ where $x \notin Y$. Choose a sequence (y_n) in Y such that $(y_n) \rightarrow x$ (which can be done since $x \in \bar{Y}$ by Theorem 2.2.A(iii)). Then (y_n) is Cauchy (by Proposition 2.9(a)) but (y_n) does not converge in Y and so Y is not complete and not a Banach space. \square

Theorem 2.20

Theorem 2.20. Extension Theorem.

Suppose that X_0 is a dense subspace of the normed linear space X such that $T_0 \in \mathcal{B}(X_0, Z)$ (i.e., T_0 is a bounded linear operator from X_0 to Z), where Z is a Banach space. Then T_0 has a unique extension to an operator $T \in \mathcal{B}(X, Z)$. Moreover, $\|T\| = \|T_0\|$, and if T_0 is an isometry, then so is T .

Proof. Let $x \in X$. Since X_0 is dense in X , then there is a Cauchy sequence $(x_n) \subseteq X_0$ convergent to x . For all $x_m, x_n \in (x_n)$ we have that $\|T_0x_m - T_0x_n\| = \|T_0(x_m - x_n)\| \leq \|T_0\| \|x_m - x_n\|$ by Note 2.4.A. Since (x_n) is Cauchy, then $(T_0x_n) \subseteq Z$ is Cauchy. Since Z is a Banach space, then Z is complete and so (T_0x_n) converges to some element of Z . Define this limit to be Tx : $Tx = \lim T_0x_n$.

We now complete the proof in 6 steps.

Theorem 2.20 (continued 1)

Proof (continued).

(i) T is well defined. That is, the definition of Tx is independent of sequence (x_n) . Suppose $(y_n) \subseteq X_0$ is convergent to x . Then $\|T_0x_n - T_0y_n\| \leq \|T_0\| \|x_n - y_n\|$. Since $(x_n) \rightarrow x$ and $(y_n) \rightarrow x$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\|x_n - y_n\| = \|x_n - x + x - y_n\| \leq \|x_n - x\| + \|x - y_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_n - y_n) \rightarrow 0$ and hence $(T_0x_n - T_0y_n) \rightarrow 0$. That is (T_0x_n) and (T_0y_n) have the same limit and so Tx is well defined.

Theorem 2.20 (continued 2)

Proof (continued).

(ii) T extends T_0 . (That is, $Tx = T_0x$ for $x \in X_0$.) For any $x \in X_0$, take the constant sequence $(x) \subseteq X_0$. Then $Tx = \lim T_0x = T_0x$.

(iii) T is linear. For any $x, y \in X$, let $(x_n), (y_n) \subseteq X_0$ be sequences converging to x and y , respectively. Then $(x_n + y_n) \rightarrow x + y$, so

$$\begin{aligned} T(x + y) &= \lim T_0(x_n + y_n) \\ &= \lim(T_0x_n + T_0y_n) \text{ since } T_0 \text{ is linear} \\ &= \lim(T_0x_n) + \lim(T_0y_n) \\ &= Tx + Ty. \end{aligned}$$

Similarly, for $\alpha \in \mathbb{R}$, since T_0 is linear, we have

$$T(\alpha x) = \lim T_0(\alpha x_n) = \lim \alpha T_0(x_n) = \alpha \lim T_0(x_n) = \alpha Tx.$$

Theorem 2.20 (continued 3)

Proof (continued).

(iv) $\|T\| = \|T_0\|$. Now $\|T\| \geq \|T_0\|$ since T is defined on X and T_0 is defined on X_0 where $X \supset X_0$. Next, for $x \in X$ where $\|x\| = 1$, choose $(x_n) \subseteq X_0$ convergent to x . Then

$$\begin{aligned} \|Tx\| &= \|\lim(T_0x_n)\| \\ &= \lim\|T_0x_n\| \text{ by Theorem 2.3(c)} \\ &\leq \lim\|T_0\|\|x_n\| \text{ by Note 2.4.A} \\ &= \|T_0\| \lim\|x_n\| \\ &= \|T_0\|\|x\| \text{ by Theorem 2.3(c)} \\ &= \|T_0\|. \end{aligned}$$

So $\|T\| \leq \|T_0\|$ and the result follows.

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Theorem 2.20 (continued 5)

Proof (continued).

(vi) T is the unique extension of T_0 . Suppose $T_1x = Tx$ for all $x \in X_0$. Let $x \in X \setminus X_0$. Then some $(x_n) \subseteq X_0$ is convergent to x since X_0 is dense in X . Then

$$\begin{aligned} T_1x &= T_1(\lim x_n) \\ &= \lim(T_1x_n) \text{ by Theorem 2.6 since } T_1 \text{ is bounded by (iv)} \\ &= \lim(Tx_n) \\ &= T(\lim x_n) \text{ by Theorem 2.6 since } T \text{ is bounded by (iv)} \\ &= Tx. \end{aligned}$$

So $T_1 = T$ on X and T is unique.

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Theorem 2.20 (continued 4)

Proof (continued).

(v) If T_0 is an isometry, then so is T . Let $x \in X$ and $(x_n) \subseteq X_0$ convergent to x . Then

$$\begin{aligned} \|Tx\| &= \|\lim(T_0x_n)\| \\ &= \lim\|T_0x_n\| \text{ by Theorem 2.3(c)} \\ &= \lim\|x_n\| \text{ since } T_0 \text{ is an isometry} \\ &= \|x\| \text{ by Theorem 2.3(c)}. \end{aligned}$$

So T is an isometry.

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Theorem 2.20 (continued 6)

Theorem 2.20. Extension Theorem.

Suppose that X_0 is a dense subspace of the normed linear space X such that $T_0 \in \mathcal{B}(X_0, Z)$ (i.e., T_0 is a bounded linear operator from X_0 to Z), where Z is a Banach space. Then T_0 has a unique extension to an operator $T \in \mathcal{B}(X, Z)$. Moreover, $\|T\| = \|T_0\|$, and if T_0 is an isometry, then so is T .

Proof (conclusion). For $x \in X$, define $Tx = \lim(T_0x_n)$ for a sequence $(x_n) \subseteq X_0$ with $(x_n) \rightarrow x$. We have shown that:

- (i) T is well defined.
- (ii) T extends T_0 .
- (iii) T is linear.
- (iv) $\|T\| = \|T_0\|$.
- (v) If T_0 is an isometry, then so is T .
- (vi) T is the unique extension of T_0 .

Then T is the unique extension of T_0 from X_0 to $T \in \mathcal{B}(X, Z)$, $\|T\| = \|T_0\|$, and if T_0 is an isometry then so is T , as claimed. \square

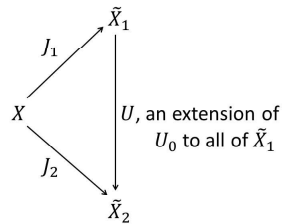
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Theorem 2.22

Theorem 2.22. Completion Theorem.

For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If (\tilde{X}_1, J_1) and (\tilde{X}_2, J_2) are completions of X , there is a surjective (onto) isometry $U : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $UJ_1 = J_2$.

Proof of Uniqueness. Notice that if J is an isometry and $x \neq y$ then $\|x - y\| > 0$ and $\|Jx - Jy\| = \|J(x - y)\| = \|x - y\| \neq 0$. So J is an injection (one to one) and J^{-1} exists. Consider $U_0 = J_2 J_1^{-1}$ mapping $J_1(X)$ into X_2 . Since J_1 is one to one, $J_1^{-1}(J_1(X)) = X$, and so U_0 maps $J_1(X)$ onto $J_2(X)$:



Theorem 2.22 (continued)

Theorem 2.22. Completion Theorem.

For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If (\tilde{X}_1, J_1) and (\tilde{X}_2, J_2) are completions of X , there is a surjective (onto) isometry $U : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $UJ_1 = J_2$.

Proof (continued). Since J_1 and J_2 are isometries, then $U_0 = J_2 J_1^{-1}$ is an isometry. Since $J_2(X)$ is dense in \tilde{X}_2 (by the definition of completion), then by Theorem 2.20 U_0 extends to an isometry U from \tilde{X}_1 to \tilde{X}_2 (notice that U_0 is bounded since it is an isometry and $\|U_0\| = 1$). Since \tilde{X}_1 is complete, the isometric image $U(\tilde{X}_1)$ is complete (images of Cauchy sequences are Cauchy with corresponding limits). So $U(\tilde{X}_1)$ is closed in \tilde{X}_2 by Theorem 2.16. Since $U(\tilde{X}_1)$ contains $J_2(X)$, $U(\tilde{X}_1)$ is dense in \tilde{X}_2 . A dense closed subset must be all of the set. That is, $U(\tilde{X}_1) = \tilde{X}_2$ where $UJ_1 = J_2$ and hence the completion of X is unique. \square