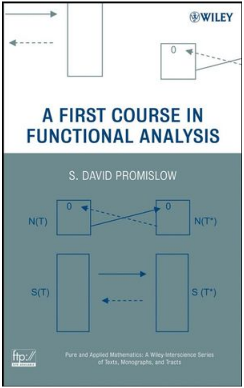
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.5. Completeness—Proofs of Theorems



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Proposition 2.9

Proposition 2.9. In a normed linear space:

(a) A convergent sequence is Cauchy.

(b) A Cauchy sequence (x_n) is bounded. That is, there is a $k > 0$ such that $\|x_n\| < k$ for all n.

(c) All subsequences of a Cauchy sequence are Cauchy.

(d) If (x_n) is Cauchy and some subsequence converges to x, then (x_n) converges to x.

Proof. (a). Let (x_n) be convergent with limit x. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|x - x_n\| < \varepsilon/2$. So let $m, n \geq N$. Then

$$\|x_m - x_n\| = \|x_m - x + x - x_n\| \leq \|x_m - x\| + \|x_n - x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So (x_n) is Cauchy, as claimed.

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So (x_n) is Cauchy, as claimed.

Proposition 2.9 (continued 1)

Proposition 2.9(b). In a normed linear space:

(b) A Cauchy sequence (x_n) is bounded. That is, there is a $k > 0$ such that $\|x_n\| < k$ for all n.

Proof (continued). (b). Suppose (x_n) is Cauchy. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|x_n - x_m\| < \varepsilon = 1$. In particular, with $n = N$ we have $\|x_N - x_m\| < \varepsilon = 1$ and so by the (Backwards) Triangle Inequality $\|x_m\| < \|x_N\| + 1$. Let

$$k = \max\{\|x_1\|, \|x_2\|, \|x_3\|, \ldots, \|x_{N-1}\|, \|x_N\| + 1\}.$$

Then for all $n \in \mathbb{N}$, $\|x_n\| \leq k$ and so $\{x_n\}$ is bounded, as claimed.

Proposition 2.9 (continued 2)

Proposition 2.9(c). In a normed linear space:

(c) All subsequences of a Cauchy sequence are Cauchy.

Proof (continued). (c). Let (x_{n_k}) be a subsequence of Cauchy sequence (x_n). Since (x_n) is Cauchy then, by definition, for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_n - x_m\| < \varepsilon$. For all $k, \ell \geq N$ we have $n_k \geq k \geq N$ (notice that $n_k \geq k$) and $n_\ell \geq \ell \geq N$, so that $\|x_{n_k} - x_{n_\ell}\| < \varepsilon$. Hence $\{x_{n_k}\}$ is Cauchy, as claimed.

Proposition 2.9 (continued 3)

Proposition 2.9(d). In a normed linear space:

(d) If (x_n) is Cauchy and some subsequence converges to x, then (x_n) converges to x.

Proof (continued). (d). Let (x_n) be Cauchy. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $\|x_n - x_m\| < \varepsilon/2$. Let (x_{n_k}) converge to x. Then there exists $J \in \mathbb{N}$ (where we take, WLOG, $J \geq N$) such that for all $k \geq J$, we have $\|x_{n_k} - x\| \leq \varepsilon/2$ (notice that $n_k \geq k$, so that $n_J \geq J$). In particular, $\|x_{n_J} - x\| \leq \varepsilon/2$. So for $n \geq J \geq N$, we have

$$\|x_n - x\| = \|x_n - x_{n_J} + x_{n_J} - x\| \leq \|x_n - x_{n_J}\| + \|x_{n_J} - x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $(x_n) \to x$, as claimed. $\square$

Proposition 2.10

Proposition 2.10.

(a) A fast Cauchy sequence is Cauchy.

(b) Any Cauchy sequence contains a fast Cauchy subsequence.

Proof. (a). Let (x_n) be fast Cauchy and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/2^{N-1} < \varepsilon$. Then for $n > m \geq N$ we have

$$\|x_n - x_m\| \leq \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\|$$

$$\leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \leq \sum_{k=m}^{n-1} \frac{1}{2^k} < \sum_{k=m}^{\infty} \frac{1}{2^k}$$

$$= \frac{1/2^m}{1 - 1/2} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \varepsilon.$$

So the sequence is Cauchy, as claimed.

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So the sequence is Cauchy, as claimed.

Proposition 2.10 (continued 1)

Proposition 2.10.

(b) Any Cauchy sequence contains a fast Cauchy subsequence.

Proof (continued). (b) Let (x_n) be Cauchy.

Choose $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$ we have $\|x_m - x_n\| < 1/2$.

Choose N_2 such that $N_2 > N_1$ and for all $m, n \geq N_2$ we have $\|x_m - x_n\| < 1/2^2$.

Now inductively choose $N_k > N_{k-1}$ such that for all $m, n \geq N_k$ we have $\|x_m - x_n\| < 1/2^k$.

Now consider the subsequence (x_{N_k}). Then for any $k \in \mathbb{N}$ we have $\|x_{N_{k+1}} - x_{N_k}\| \leq 1/2^k$ since $N_{k+1} > N_k \geq N_k$. Therefore (x_{N_k}) is fast Cauchy, as claimed. □

Proposition 2.11

Proposition 2.11. If $x = \sum_{i=1}^{\infty} x_i$ exists, then $\|x\| \leq \sum_{i=1}^{\infty} \|x_i\|$.

Proof. By continuity of the norm (Theorem 2.3(c)) we have (from the Triangle Inequality)

$$\|x\| = \left\|\lim_{n\to\infty} s_n\right\| = \lim_{n\to\infty} \|s_n\| = \lim_{n\to\infty} \left\|\sum_{i=1}^{n} x_i\right\|$$

$$\leq \lim_{n\to\infty} \left(\sum_{i=1}^{n} \|x_i\|\right) = \sum_{i=1}^{\infty} \|x_i\|.$$

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□

Theorem 2.12

Theorem 2.12. A normed linear space X is complete if and only if every absolutely convergent series is convergent.

Proof. Suppose X is complete and (x_i) yields an absolutely convergent series. Then $\sum_{i=1}^{\infty} \|x_i\| < \infty$. Let $\varepsilon > 0$. Now for $N \in \mathbb{N}$ sufficiently large, $\sum_{i=N}^{\infty} \|x_i\| < \varepsilon$. So for $n > m \geq N$ we have for the partial sums

$$\|s_n - s_m\| = \left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\| \text{ by Triangle Inequality}$$

$$\leq \sum_{i=m+1}^{\infty} \|x_i\| < \varepsilon.$$

So (s_i) is Cauchy and since X is complete, (s_i) is convergent. That is, the absolutely convergent series $\sum x_i$ is convergent.

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So (s_i) is Cauchy and since X is complete, (s_i) is convergent. That is, the absolutely convergent series $\sum x_i$ is convergent.

Theorem 2.12 (continued)

Proof (continued). Now suppose that every absolutely convergent series is convergent. Let (x_n) be a fast Cauchy sequence. The series $\sum_{i=1}^{\infty}(x_{i+1} - x_i)$ is absolutely convergent:

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| \leq \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 1,$$

and so is convergent (by hypothesis), say to x. Since the partial sum for this series is $s_n = x_{n+1} - x_1$ and $s_n \to x$, we have that $x_{n+1} \to x + x_1$, or $(x_n) \to x + x_1$. So the fast Cauchy sequence is convergent. Since (x_n) is an arbitrary fast Cauchy sequence, this shows that every fast Cauchy sequence in X is convergent. Let (x'_n) be any Cauchy sequence in X. By Proposition 2.10(b) there is a subsequence (x'_{n_k}) of (x'_n) which is fast Cauchy. As shown above, (x'_{n_k}) converges in X. By Proposition 2.9(d), (x'_n) converges in X. Since (x'_n) is an arbitrary Cauchy sequence in X, then X is complete. □

Theorem 2.12 (continued)

Proof (continued). Now suppose that every absolutely convergent series is convergent. Let (x_n) be a fast Cauchy sequence. The series $\sum_{i=1}^{\infty}(x_{i+1} - x_i)$ is absolutely convergent:

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and so is convergent (by hypothesis), say to x. Since the partial sum for this series is $s_n = x_{n+1} - x_1$ and $s_n \to x$, we have that $x_{n+1} \to x + x_1$, or $(x_n) \to x + x_1$. So the fast Cauchy sequence is convergent. Since (x_n) is an arbitrary fast Cauchy sequence, this shows that every fast Cauchy sequence in X is convergent. Let (x_n') be any Cauchy sequence in X. By Proposition 2.10(b) there is a subsequence (x_{n_k}') of (x_n') which is fast Cauchy. As shown above, (x_{n_k}') converges in X. By Proposition 2.9(d), (x_n') converges in X. Since (x_n') is an arbitrary Cauchy sequence in X, then X is complete. $\square$

Lemma 2.13

Lemma 2.13. Suppose that X is a subspace of the space of all functions from set S to field $\mathbb{F}$, $F(S)$, and that $\|\cdot\|$ is a norm on X for which the closed unit ball $\overline{B}(1)$ is closed under pointwise limits. That is, if Cauchy sequence $(f_n) \subset \overline{B}(1)$ converges pointwise to f, then $f \in X$ and $f \in \overline{B}(1)$. If a sequence (f_n) in X is Cauchy and converges pointwise to f, then $f \in X$ and (f_n) converges to f with respect to $\|\cdot\|$.

Proof. Let (f_n) be a Cauchy sequence in X which converges pointwise to f. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|f_n - f_m\| \leq \varepsilon$. Since $\overline{B}(1)$ is closed under pointwise limits, then by scaling $f \in \overline{B}(1)$ to $\varepsilon f / \|f\| \in \overline{B}(\varepsilon)$, we see that $\overline{B}(\varepsilon)$ is closed under pointwise limits.

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Theorem 2.14

Theorem 2.14. The space of all bounded functions from set S to field $\mathbb{F}$ (taken to be $\mathbb{R}$ or $\mathbb{C}$), $B(S)$, is complete with respect to the sup norm.

Proof. Let (f_n) be a Cauchy sequence in $B(S)$. For any point $s \in S$,

$$|f_m(s) - f_n(s)| \leq \sup_{s \in S} |f_m(s) - f_n(s)| = \|f_m - f_n\|.$$

So the sequence $(f_n(s))$ is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete, $(f_n(s))$ converges to some point in $\mathbb{F}$, denoted $f(s)$. So f is the pointwise limit of (f_n) on S.

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So the sequence $(f_n(s))$ is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete, $(f_n(s))$ converges to some point in $\mathbb{F}$, denoted $f(s)$. So f is the pointwise limit of (f_n) on S. Moreover, if $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$ (or equivalently, $f_n \in \overline{B}(1)$) then for any $s \in S$ we have $|f_n(s)| \leq \|f_n\| \leq 1$, and so $|f(s)| = |\lim_{n\to\infty} f_n(s)| = \lim_{n\to\infty} |f_n(s)| \leq 1$. Therefore $f \in \overline{B}(1)$. So by Lemma 2.13, $(f_n) \to f$ with respect to the sup norm. □

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So the sequence $(f_n(s))$ is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete, $(f_n(s))$ converges to some point in $\mathbb{F}$, denoted $f(s)$. So f is the pointwise limit of (f_n) on S. Moreover, if $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$ (or equivalently, $f_n \in \overline{B}(1)$) then for any $s \in S$ we have $|f_n(s)| \leq \|f_n\| \leq 1$, and so $|f(s)| = |\lim_{n\to\infty} f_n(s)| = \lim_{n\to\infty} |f_n(s)| \leq 1$. Therefore $f \in \overline{B}(1)$. So by Lemma 2.13, $(f_n) \to f$ with respect to the sup norm. □

Theorem 2.16

Theorem 2.16. A subspace Y of a Banach space X is itself a Banach space if and only if Y is closed.

Proof. First, suppose Y is closed and let (y_n) be a Cauchy sequence in Y. Then (y_n) is a Cauchy sequence in X and since X is a Banach space, $(y_n) \to x$ for some $x \in X$. Since Y is closed then, by Theorem 2.2.A(iii), $x \in Y$ and so (y_n) is convergent in Y and Y is a Banach space.

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Conversely, suppose Y is not closed. Then there is $x \in \overline{Y}$ where $x \notin Y$. Choose a sequence (y_n) in Y such that $(y_n) \to x$ (which can be done since $x \in \overline{Y}$ by Theorem 2.2.A(iii)). Then (y_n) is Cauchy (by Proposition 2.9(a)) but (y_n) does not converge in Y and so Y is not complete and not a Banach space. □

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Theorem 2.20

Theorem 2.20. Extension Theorem.
Suppose that X_0 is a dense subspace of the normed linear space X such that $T_0 \in \mathcal{B}(X_0, Z)$ (i.e., T_0 is a bounded linear operator from X_0 to Z), where Z is a Banach space. Then T_0 has a unique extension to an operator $T \in \mathcal{B}(X, Z)$. Moreover, $\|T\| = \|T_0\|$, and if T_0 is an isometry, then so is T.

Proof. Let $x \in X$. Since X_0 is dense in X, then there is a Cauchy sequence $(x_n) \subseteq X_0$ convergent to x. For all $x_m, x_n \in (x_n)$ we have that $\|T_0 x_m - T_0 x_n\| = \|T_0(x_m - x_n)\| \leq \|T_0\| \|x_m - x_n\|$ by Note 2.4.A. Since (x_n) is Cauchy, then $(T_0 x_n) \subseteq Z$ is Cauchy. Since Z is a Banach space, then Z is complete and so $(T_0 x_n)$ converges to some element of Z. Define this limit to be Tx: $Tx = \lim T_0 x_n$.

We now complete the proof in 6 steps.

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Proof. Let $x \in X$. Since X_0 is dense in X, then there is a Cauchy sequence $(x_n) \subseteq X_0$ convergent to x. For all $x_m, x_n \in (x_n)$ we have that $\|T_0x_m - T_0x_n\| = \|T_0(x_m - x_n)\| \leq \|T_0\|\|x_m - x_n\|$ by Note 2.4.A. Since (x_n) is Cauchy, then $(T_0x_n) \subseteq Z$ is Cauchy. Since Z is a Banach space, then Z is complete and so (T_0x_n) converges to some element of Z. Define this limit to be Tx: $Tx = \lim T_0x_n$.

We now complete the proof in 6 steps.

Theorem 2.20 (continued 1)

Proof (continued).

(i) T is well defined. That is, the definition of Tx is independent of sequence (x_n). Suppose $(y_n) \subseteq X_0$ is convergent to x. Then $\|T_0x_n - T_0y_n\| \le \|T_0\|\|x_n - y_n\|$. Since $(x_n) \to x$ and $(y_n) \to x$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \ge N$

$$\|x_n - y_n\| = \|x_n - x + x - y_n\| \le \|x_n - x\| + \|x - y_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_n - y_n) \to 0$ and hence $(T_0x_n - T_0y_n) \to 0$. That is (T_0x_n) and (T_0y_n) have the same limit and so Tx is well defined.

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(i) T is well defined. That is, the definition of Tx is independent of sequence (x_n). Suppose $(y_n) \subseteq X_0$ is convergent to x. Then $\|T_0 x_n - T_0 y_n\| \leq \|T_0\| \|x_n - y_n\|$. Since $(x_n) \to x$ and $(y_n) \to x$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\|x_n - y_n\| = \|x_n - x + x - y_n\| \leq \|x_n - x\| + \|x - y_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_n - y_n) \to 0$ and hence $(T_0 x_n - T_0 y_n) \to 0$. That is $(T_0 x_n)$ and $(T_0 y_n)$ have the same limit and so Tx is well defined.

Theorem 2.20 (continued 2)

Proof (continued).
(ii) T **extends** T_0**.** (That is, $Tx = T_0 x$ for $x \in X_0$.) For any $x \in X_0$, take the constant sequence $(x) \subset X_0$. Then $Tx = \lim T_0 x = T_0 x$.

(iii) T **is linear.** For any $x, y \in X$, let $(x_n), (y_n) \subseteq X_0$ be sequences converging to x and y, respectively. Then $(x_n + y_n) \to x + y$, so

$$\begin{aligned} T(x+y) &= \lim T_0(x_n + y_n) \\ &= \lim(T_0 x_n + T_0 y_n) \text{ since } T_0 \text{ is linear} \\ &= \lim(T_0 x_n) + \lim(T_0 y_n) \\ &= Tx + Ty. \end{aligned}$$

Similarly, for $\alpha \in \mathbb{R}$, since T_0 is linear, we have

$$T(\alpha x) = \lim T_0(\alpha x_n) = \lim \alpha T_0(x_n) = \alpha \lim T_0(x_n) = \alpha T(x).$$

Theorem 2.20 (continued 2)

Proof (continued).
(ii) T **extends** T_0**.** (That is, $Tx = T_0 x$ for $x \in X_0$.) For any $x \in X_0$, take the constant sequence $(x) \subset X_0$. Then $Tx = \lim T_0 x = T_0 x$.

(iii) T **is linear.** For any $x, y \in X$, let $(x_n), (y_n) \subseteq X_0$ be sequences converging to x and y, respectively. Then $(x_n + y_n) \to x + y$, so

$$\begin{aligned} T(x+y) &= \lim T_0(x_n + y_n) \\ &= \lim(T_0 x_n + T_0 y_n) \text{ since } T_0 \text{ is linear} \\ &= \lim(T_0 x_n) + \lim(T_0 y_n) \\ &= Tx + Ty. \end{aligned}$$

Similarly, for $\alpha \in \mathbb{R}$, since T_0 is linear, we have

$$T(\alpha x) = \lim T_0(\alpha x_n) = \lim \alpha T_0(x_n) = \alpha \lim T_0(x_n) = \alpha T(x).$$

Theorem 2.20 (continued 3)

Proof (continued).

(iv) $\|T\| = \|T_0\|$. Now $\|T\| \geq \|T_0\|$ since T is defined on X and T_0 is defined on X_0 where $X \supset X_0$. Next, for $x \in X$ where $\|x\| = 1$, choose $(x_n) \subseteq X_0$ convergent to x. Then

$$\begin{aligned}
\|Tx\| &= \|\lim(T_0 x_n)\| \\
&= \lim \|T_0 x_n\| \text{ by Theorem 2.3(c)} \\
&\leq \lim \|T_0\| \|x_n\| \text{ by Note 2.4.A} \\
&= \|T_0\| \lim \|x_n\| \\
&= \|T_0\| \|x\| \text{ by Theorem 2.3(c)} \\
&= \|T_0\|.
\end{aligned}$$

So $\|T\| \leq \|T_0\|$ and the result follows.

Theorem 2.20 (continued 4)

Proof (continued).
(v) If T_0 is an isometry, then so is T. Let $x \in X$ and $(x_n) \subseteq X_0$ convergent to x. Then

$$\begin{aligned} \|Tx\| &= \|\lim(T_0 x_n)\| \\ &= \lim \|T_0 x_n\| \text{ by Theorem 2.3(c)} \\ &= \lim \|x_n\| \text{ since } T_0 \text{ is an isometry} \\ &= \|x\| \text{ by Theorem 2.3(c).} \end{aligned}$$

So T is an isometry.

Theorem 2.20 (continued 5)

Proof (continued).
(vi) T is the unique extension of T_0. Suppose $T_1 x = Tx$ for all $x \in X_0$. Let $x \in X \setminus X_0$. Then some $(x_n) \subseteq X_0$ is convergent to x since X_0 is dense in X. Then

$$\begin{aligned} T_1 x &= T_1(\lim x_n) \\ &= \lim(T_1 x_n) \text{ by Theorem 2.6 since } T_1 \text{ is bounded by (iv)} \\ &= \lim(T x_n) \\ &= T(\lim x_n) \text{ by Theorem 2.6 since } T \text{ is bounded by (iv)} \\ &= Tx. \end{aligned}$$

So $T_1 = T$ on X and T is unique.

Theorem 2.20 (continued 6)

Theorem 2.20. Extension Theorem.
Suppose that X_0 is a dense subspace of the normed linear space X such that $T_0 \in \mathcal{B}(X_0, Z)$ (i.e., T_0 is a bounded linear operator from X_0 to Z), where Z is a Banach space. Then T_0 has a unique extension to an operator $T \in \mathcal{B}(X, Z)$. Moreover, $\|T\| = \|T_0\|$, and if T_0 is an isometry, then so is T.

Proof (conclusion). For $x \in X$, define $Tx = \lim(T_0 x_n)$ for a sequence $(x_n) \subseteq X_0$ with $(x_n) \to n$. We have shown that:

(i) T is well defined.
(ii) T extends T_0.
(iii) T is linear.
(iv) $\|T\| = \|T_0\|$.
(v) If T_0 is an isometry, then so is T.
(vi) T is the unique extension of T_0.

Then T is the unique extension of T_0 from X_0 to $T \in \mathcal{B}(X, Z)$, $\|T\| = \|T_0\|$, and if T_0 is an isometry then so is T, as claimed. □

Theorem 2.22

Theorem 2.22. Completion Theorem.
For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If $(\tilde{X}_1, J_1)$ and $(\tilde{X}_2, J_2)$ are completions of X, there is a surjective (onto) isometry $U : \tilde{X}_1 \to \tilde{X}_2$ such that $UJ_1 = J_2$.

Proof of Uniqueness. Notice that if J is an isometry and $x \neq y$ then $\|x - y\| > 0$ and $\|Jx - Jy\| = \|J(x - y)\| = \|x - y\| \neq 0$. So J is an injection (one to one) and J^{-1} exists. Consider $U_0 = J_2 J_1^{-1}$ mapping $J_1(X)$ into X_2. Since J_1 is one to one, $J_1^{-1}(J_1(X)) = X$, and so U_0 maps $J_1(X)$ onto $J_2(X)$:

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$$\begin{tikzcd}
 & \tilde{X}_1 \arrow[dd, "U\text{, an extension of } U_0 \text{ to all of } \tilde{X}_1"] \\
X \arrow[ur, "J_1"] \arrow[dr, "J_2"'] & \\
 & \tilde{X}_2
\end{tikzcd}$$

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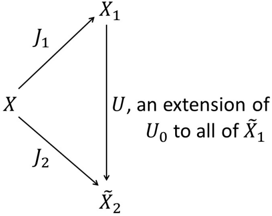

Theorem 2.22 (continued)

Theorem 2.22. Completion Theorem.
For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If $(\tilde{X}_1, J_1)$ and $(\tilde{X}_2, J_2)$ are completions of X, there is a surjective (onto) isometry $U : \tilde{X}_1 \to \tilde{X}_2$ such that $UJ_1 = J_2$.

Proof (continued). Since J_1 and J_2 are isometries, then $U_0 = J_2 J_1^{-1}$ is an isometry. Since $J_2(X)$ is dense in $\tilde{X}_2$ (by the definition of completion), then by Theorem 2.20 U_0 extends to an isometry U from $\tilde{X}_1$ to $\tilde{X}_2$ (notice that U_0 is bounded since it is an isometry and $\|U_0\| = 1$). Since $\tilde{X}_1$ is complete, the isometric image $U(\tilde{X}_1)$ is complete (images of Cauchy sequences are Cauchy with corresponding limits). So $U(\tilde{X}_1)$ is closed in $\tilde{X}_2$ by Theorem 2.16. Since $U(\tilde{X}_1)$ contains $J_2(X)$, $U(\tilde{X}_1)$ is dense in $\tilde{X}_2$. A dense closed subset must be all of the set. That is, $U(\tilde{X}_1) = \tilde{X}_2$ where $UJ_1 = J_2$ and hence the completion of X is unique. □

Theorem 2.22 (continued)

Theorem 2.22. Completion Theorem.
For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If $(\tilde{X}_1, J_1)$ and $(\tilde{X}_2, J_2)$ are completions of X, there is a surjective (onto) isometry $U : \tilde{X}_1 \to \tilde{X}_2$ such that $UJ_1 = J_2$.

Proof (continued). Since J_1 and J_2 are isometries, then $U_0 = J_2 J_1^{-1}$ is an isometry. Since $J_2(X)$ is dense in $\tilde{X}_2$ (by the definition of completion), then by Theorem 2.20 U_0 extends to an isometry U from $\tilde{X}_1$ to $\tilde{X}_2$ (notice that U_0 is bounded since it is an isometry and $\|U_0\| = 1$). Since $\tilde{X}_1$ is complete, the isometric image $U(\tilde{X}_1)$ is complete (images of Cauchy sequences are Cauchy with corresponding limits). So $U(\tilde{X}_1)$ is closed in $\tilde{X}_2$ by Theorem 2.16. Since $U(\tilde{X}_1)$ contains $J_2(X)$, $U(\tilde{X}_1)$ is dense in $\tilde{X}_2$. A dense closed subset must be all of the set. That is, $U(\tilde{X}_1) = \tilde{X}_2$ where $UJ_1 = J_2$ and hence the completion of X is unique. $\square$