Chapter 2. Normed Linear Spaces: The Basics

2.6. Comparisons of Norms—Proofs of Theorems
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Proposition 2.23. \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \) if and only if there is \( K > 0 \) such that \( \| x \|_1 \leq K \| x \|_2 \) for all \( x \in X \).

Proof. Suppose such a \( K \) exists and let \( (x_n) \) be a sequence converging to \( x \) with respect to \( \| \cdot \|_2 \). Then for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for \( n \geq N \) we have \( \| x_n - x \|_2 < \varepsilon / K \).
Proposition 2.23. \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \) if and only if there is \( K > 0 \) such that \( \|x\|_1 \leq K \|x\|_2 \) for all \( x \in X \).

Proof. Suppose such a \( K \) exists and let \((x_n)\) be a sequence converging to \( x \) with respect to \( \| \cdot \|_2 \). Then for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for \( n \geq N \) we have \( \|x_n - x\|_2 < \varepsilon/K \). So for all \( n \geq N \) we also have

\[
\|x_n - x\|_1 \leq K \|x_n - x\|_2 < K \left( \frac{\varepsilon}{K} \right) = \varepsilon.
\]

So \((x_n) \to x\) with respect to \( \| \cdot \|_1 \) and \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \).
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So \( (x_n) \rightarrow x \) with respect to \( \| \cdot \|_1 \) and \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \).
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Proof (continued). Suppose \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \). Define \( T : (X, \| \cdot \|_2) \to (X, \| \cdot \|_1) \) as \( Tx = x \) for all \( x \in X \).
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Theorem 2.23 (continued)

**Proposition 2.23.** \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \) if and only if there is \( K > 0 \) such that \( \| x \|_1 \leq K \| x \|_2 \) for all \( x \in X \).

**Proof (continued).** Suppose \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \). Define \( T : (X, \| \cdot \|_2) \to (X, \| \cdot \|_1) \) as \( Tx = x \) for all \( x \in X \). Then for \( (x_n) \to x \) in \( (X, \| \cdot \|_2) \), since \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \), we have \( (x_n) \to x \) in \( (X, \| \cdot \|_1) \); that is, \( (Tx_n) \to Tx \) in \( (X, \| \cdot \|_1) \). Since \( (x_n) \) is an arbitrary convergent sequence in \( (X, \| \cdot \|_2) \) then, by Theorem 2.1.A, \( T \) is continuous. Since \( T \) is continuous, by Theorem 2.6, \( T \) is bounded. Let \( K = \| T \| \). We then have, for all \( x \in X \),

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\| x \|_1 = \| Tx \|_1 \leq \| T \| \| x \|_2 = K \| x \|_2,
\]

as claimed.
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Proof (continued). Suppose \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \). Define \( T : (X, \| \cdot \|_2) \to (X, \| \cdot \|_1) \) as \( Tx = x \) for all \( x \in X \). Then for \( (x_n) \to x \) in \( (X, \| \cdot \|_2) \), since \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \), we have \( (x_n) \to x \) in \( (X, \| \cdot \|_1) \); that is, \( (Tx_n) \to Tx \) in \( (X, \| \cdot \|_1) \). Since \( (x_n) \) is an arbitrary convergent sequence in \( (X, \| \cdot \|_2) \) then, by Theorem 2.1.A, \( T \) is continuous. Since \( T \) is continuous, by Theorem 2.6, \( T \) is bounded. Let \( K = \| T \| \). We then have, for all \( x \in X \),

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Proposition 2.24. $\| \cdot \|_1$ is weaker than $\| \cdot \|_2$ if and only if every $\| \cdot \|_1$ open ball contains a $\| \cdot \|_2$ open ball.

Proof of “only if” part. Suppose $\| \cdot \|_1$ is weaker than $\| \cdot \|_2$. Then by Proposition 2.23, there exists $K > 0$ such that $\|x\|_1 \leq K \|x\|_2$ for all $x \in X$. 
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**Proposition 2.24.** \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \) if and only if every \( \| \cdot \|_1 \) open ball contains a \( \| \cdot \|_2 \) open ball.

**Proof of “only if” part.** Suppose \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \). Then by Proposition 2.23, there exists \( K > 0 \) such that \( \| x \|_1 \leq K \| x \|_2 \) for all \( x \in X \). Then \( B(x; r) = \{ y \in X \mid \| x - y \|_1 < r \} \) contains

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B(x; r/K) = \{ y \in X \mid \| x - y \|_2 < r/K \},
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since \( \| x - y \|_2 < r/K \) implies that

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**Theorem 2.26.** If $X$ is a Banach space with respect to a norm $\| \cdot \|_1$, it is also a Banach space with respect to any equivalent norm.

**Proof.** We need only show that a Cauchy sequence convergent under $\| \cdot \|_1$ is convergent under an equivalent norm, say $\| \cdot \|_2$. 
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**Proof.** We need only show that a Cauchy sequence convergent under $\| \cdot \|_1$ is convergent under an equivalent norm, say $\| \cdot \|_2$. Let $(x_n)$ be Cauchy with respect to $\| \cdot \|_2$. By Proposition 2.23, there is $K \geq 0$ such that $\| x \|_1 \leq K \| x \|_2$ for all $x \in X$. So $(x_n)$ is Cauchy with respect to $\| \cdot \|_1$ (use $K \varepsilon$ in the definition of Cauchy with respect to $\| \cdot \|_2$).
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**Proof.** We need only show that a Cauchy sequence convergent under $\| \cdot \|_1$ is convergent under an equivalent norm, say $\| \cdot \|_2$. Let $(x_n)$ be Cauchy with respect to $\| \cdot \|_2$. By Proposition 2.23, there is $K \geq 0$ such that $\|x\|_1 \leq K \|x\|_2$ for all $x \in X$. So $(x_n)$ is Cauchy with respect to $\| \cdot \|_1$ (use $K \varepsilon$ in the definition of Cauchy with respect to $\| \cdot \|_2$). Since $(X, \| \cdot \|_1)$ is a Banach space, then $(x_n)$ converges to some $x$. Since $\| \cdot \|_2$ weaker than $\| \cdot \|_1$ (equivalent, actually), then $(x_n)$ also converges with respect to $\| \cdot \|_2$. Therefore $(X, \| \cdot \|_2)$ is a Banach space. \qed