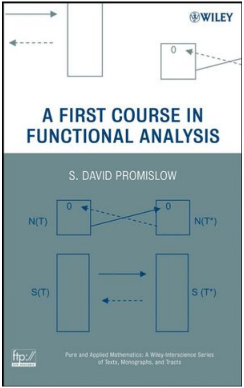
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.6. Comparisons of Norms—Proofs of Theorems



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Theorem 2.23

Proposition 2.23. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if there is $K > 0$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$.

Proof. Suppose such a K exists and let (x_n) be a sequence converging to x with respect to $\|\cdot\|_2$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $\|x_n - x\|_2 < \varepsilon/K$. So for all $n \geq N$ we also have

$$\|x_n - x\|_1 \leq K\|x_n - x\|_2 < K\left(\frac{\varepsilon}{K}\right) = \varepsilon.$$

So $(x_n) \to x$ with respect to $\|\cdot\|_1$ and $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$.

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Theorem 2.23 (continued)

Proposition 2.23. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if there is $K > 0$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$.

Proof (continued). Suppose $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. Define $T : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ as $Tx = x$ for all $x \in X$. Then for $(x_n) \to x$ in $(X, \|\cdot\|_2)$, since $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$, we have $(x_n) \to x$ in $(X, \|\cdot\|_1)$; that is, $(Tx_n) \to Tx$ in $(X, \|\cdot\|_1)$. Since (x_n) is an arbitrary convergent sequence in $(X, \|\cdot\|_2)$ then, by Theorem 2.1.A, T is continuous. Since T is continuous, by Theorem 2.6, T is bounded. Let $K = \|T\|$. We then have by Note 2.4.A, for all $x \in X$,

$$\|x\|_1 = \|Tx\|_1 \leq \|T\|\|x\|_2 = K\|x\|_2,$$

as claimed. □

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Theorem 2.24

Proposition 2.24. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if every $\|\cdot\|_1$ open ball contains a $\|\cdot\|_2$ open ball.

Proof of "only if" part. Suppose $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. Then by Proposition 2.23, there exists $K > 0$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$. Then $B(x; r) = \{y \in X \mid \|x - y\|_1 < r\}$ contains

$$B(x; r/K) = \{y \in X \mid \|x - y\|_2 < r/K\},$$

since $\|x - y\|_2 < r/K$ implies that

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Proposition 2.26

Theorem 2.26. If X is a Banach space with respect to a norm $\| \cdot \|_1$, it is also a Banach space with respect to any equivalent norm.

Proof. We need only show that a Cauchy sequence convergent under $\| \cdot \|_1$ is convergent under an equivalent norm, say $\| \cdot \|_2$. Let (x_n) be Cauchy with respect to $\| \cdot \|_2$. By Proposition 2.23, there is $K > 0$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$. So (x_n) is Cauchy with respect to $\| \cdot \|_1$ (use $K\varepsilon$ in the definition of Cauchy with respect to $\| \cdot \|_2$). Since $(X, \| \cdot \|_1)$ is a Banach space, then (x_n) converges to some x. Since $\| \cdot \|_2$ weaker than $\| \cdot \|_1$ (equivalent, actually), then (x_n) also converges with respect to $\| \cdot \|_2$. Therefore $(X, \| \cdot \|_2)$ is a Banach space. $\square$

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