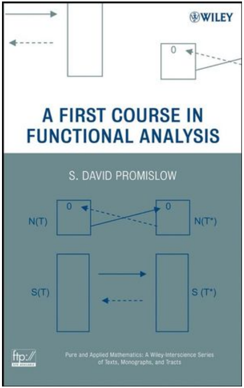
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.7. Quotient Spaces—Proofs of Theorems



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Theorem 2.27. Let N be a closed subspace of the normed linear space X.

(a) The quantity $\|\overline{x}\|$ defines a norm on X/N.

(b) If X is a Banach space, then X/N is a Banach space.

(c) $\|\pi_N\| = 1$.

(d) If $N = N(T)$ (the nullspace of bounded linear $T : X \to Y$) then the map $\tilde{T} : X/N \to Y$ defined as $\tilde{T}\overline{x} = Tx$ has the same norm as T: $\|\tilde{T}\| = \|T\|$.

Proof. Recall that for $\overline{x} \in X/N$, we define the (alleged) norm on X/N as

$$\|\overline{x}\| = \inf\{\|x - z\| \mid z \in N\} = d(x, N).$$

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Theorem 2.27 (continued 1)

(a) The quantity $\|\overline{x}\|$ defines a norm on X/N.

Proof (continued). Given $x_1, x_2 \in X$ and any $\varepsilon > 0$, choose $z_1, z_2 \in N$ so that $\|x_1 - z_1\| < \|\overline{x}_1\| + \varepsilon/2$ and $\|x_2 - z_2\| < \|\overline{x}_2\| + \varepsilon/2$. Then

$$\begin{aligned} \|\overline{x}_1 + \overline{x}_2\| &= \|\overline{x_1 + x_2}\| \\ &\leq \|(x_1 + x_2) - (z_1 + z_2)\| \text{ since } \|\overline{x_1 + x_2}\| \text{ is an infimum} \\ &\leq \|x_1 - z_1\| + \|x_2 - z_2\| \text{ by the Triangle Inequality in } (X, \|\cdot\|) \\ &< \|\overline{x}_1\| + \|\overline{x}_2\| + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Triangle Inequality holds on the X/N "norm."

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Since $\varepsilon > 0$ is arbitrary, the Triangle Inequality holds on the X/N "norm." For $x \in X$ and $\alpha \in \mathbb{F}$, $\alpha \neq 0$, fixed and for any $z \in N$, we have

$$\|\alpha x - z\| = |\alpha|\|x - z/\alpha\| \geq |\alpha|\|\overline{x}\| \text{ since } z/\alpha \in N.$$

Taking an infimum over all $z \in N$ in the inequality implies that $\|\overline{\alpha x}\| \geq |\alpha|\|\overline{x}\|$ (and this also holds if $\alpha = 0$).

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Theorem 2.27 (continued 2)

Theorem 2.27. Let N be a closed subspace of the normed linear space X.
(a) The quantity $\|\overline{x}\|$ defines a norm on X/N.

Proof (continued). Given $r > 1$, choose $z_0 \in N$ such that $\|x - z_0\| \leq r\|\overline{x}\|$ (this can be done since $r\|\overline{x}\| > \|\overline{x}\|$). Then

$$\begin{aligned} \|\overline{\alpha x}\| &\leq \|\alpha x - \alpha z_0\| \text{ since } \alpha z_0 \in N \\ &= |\alpha|\|x - z_0\| \\ &\leq |\alpha| r\|\overline{x}\| \text{ since } \|x - z_0\| \leq r\|\overline{x}\|. \end{aligned}$$

Since this holds for all $r > 1$, it holds for $r = 1$ (taking a limit as $r \to 1^+$) and $\|\overline{\alpha x}\| \leq |\alpha|\|\overline{x}\|$. Combining this with the above yields $\|\overline{\alpha x}\| = |\alpha|\|\overline{x}\|$ and the scalar property holds.

Theorem 2.27 (continued 3)

Theorem 2.27. Let N be a closed subspace of the normed linear space X.
(a) The quantity $\|\overline{x}\|$ defines a norm on X/N.

Proof. (continued). Notice that $\|\overline{x}\| = 0$ if and only if

$$\inf\{\|x - z\| \mid z \in N\} = 0$$

which, in turn, holds if and only if $x \in N$ (since N is a closed linear space by hypothesis). Since N is the additive identity of X/N then $\|\overline{x}\| = 0$ if and only if $\overline{x} = N = 0$.

So $\|\cdot\|$ satisfies the definition of a norm and hence defines a norm on X/N, as claimed.

Theorem 2.27 (continued 4)

Theorem 2.27. Let N be a closed subspace of the normed linear space X.
(b) If X is a Banach space, then X/N is a Banach space.

Proof (continued). Suppose $\sum \overline{x}_i$ is an absolutely convergent series in X/N. For each $i \in \mathbb{N}$, choose $z_i \in N$ such that $\|x_i - z_i\| \le \|\overline{x}_i\| + 1/2^i$. Then

$$\sum_{i=1}^{\infty} \|x_i - z_i\| \le \sum_{i=1}^{\infty} \left(\|\overline{x}_i\| + 1/2^i\right) < \infty.$$

So $\sum(x_i - z_i)$ is absolutely convergent and, since X is a Banach space, then by Theorem 2.12, $\sum(x_i - z_i)$ is convergent. Now π_N is linear and is shown to be bounded in part (c), so is continuous by Theorem 2.6. Now $\pi_N(x_i - z_i) = \overline{x}_i$, so $\pi_N(\sum(x_i - z_i)) = \sum \pi_N(x_i - z_i) = \sum \overline{x}_i$ and $\sum \overline{x}_i$ is convergent (to $\pi_N \sum(x_i - z_i)$). By Theorem 2.12, X/N is a Banach Space, as claimed.

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Proof (continued). Suppose $\sum \overline{x}_i$ is an absolutely convergent series in X/N. For each $i \in \mathbb{N}$, choose $z_i \in N$ such that $\|x_i - z_i\| \leq \|\overline{x}_i\| + 1/2^i$. Then

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Theorem 2.27 (continued 5)

(c) $\|\pi_N\| = 1$.

Proof (continued). We have by the operator norm definition that

$$\|\pi_N\| = \sup\{\|\pi_N(x)\| \mid x \in X, \|x\| = 1\}.$$

Now $\pi_N(x) = \overline{x}$ and $\|\overline{x}\| = \inf\{\|x - z\| \mid z \in N\}$ and since N is a linear subspace, then $0 \in N$ and $\|\overline{x}\| \leq \|x\|$. With $\|x\| = 1$ we have $\|\pi_N(x)\| = \|\overline{x}\| \leq 1$ and so $\|\pi_N\| \leq 1$. Now let $\overline{x} \in X/N$ satisfy $\|\overline{x}\| = 1$. Given $r > 1$, choose $z \in N$ such that $\|x - z\| \leq r$ (this can be done by the definition of $\|\overline{x}\|$ in terms of an infimum). Then by Note 2.4.A,

$$\begin{aligned} 1 = \|\overline{x}\| &= \|\pi_N(x - z)\| \text{ since } z \in N \\ &\leq \|\pi_N\| \|x - z\| \leq \|\pi_N\| r. \end{aligned}$$

Since $r > 1$ is arbitrary, the inequality holds as $r \to 1^+$ and so $1 \leq \|\pi_N\|$. Therefore $\|\pi_N\| = 1$, as claimed.

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(d) If $N = N(T)$ (the nullspace of bounded linear $T : X \to Y$) then the map $\tilde{T} : X/N \to Y$ defined as $\tilde{T}\overline{x} = Tx$ has the same norm as T: $\|\tilde{T}\| = \|T\|$.

Proof of (d). As in the proof of part (c), if $x \in X$ satisfies $\|x\| = 1$, then $\|\overline{x}\| \leq 1$, and by Note 2.4.A

$$\|Tx\| = \|\tilde{T}\overline{x}\| \leq \|\tilde{T}\|\|\overline{x}\| \leq \|\tilde{T}\|.$$

So taking a supremum over all such x implies that $\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\} \leq \|\tilde{T}\|$. Next, given $\overline{x} \in X/N$ where $\|\overline{x}\| = 1$, for any $r > 1$ there is $z \in N$ such that $\|x - z\| \leq r$ by the definition of $\|\overline{x}\|$ in terms of infimum. Then,

$$\begin{aligned} \|\tilde{T}\overline{x}\| = \|Tx\| &= \|T(x-z)\| \text{ since } z \in N \\ &\leq \|T\|\|x-z\| \leq \|T\|r. \end{aligned}$$

Again, letting $r \to 1^+$ (and then taking suprema over all such $\overline{x}$) we get $\|\tilde{T}\| \leq \|T\|$ and the result follows. □

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