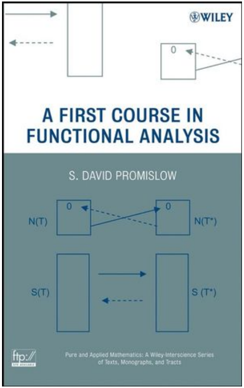
Introduction to Functional Analysis

Chapter 2. Normed Linear Spaces: The Basics

2.8. Finite Dimensional Normed Linear Spaces—Proofs of Theorems



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Proposition 2.28

Proposition 2.28. For any normed linear space Z, all elements of $\mathcal{L}(B(F), Z)$ (the set of linear operators from $B(F)$ to Z) are bounded.

Proof. For any $T \in \mathcal{L}(B(F), Z)$, let $K = \max\{\|T\delta_i\|\}$ (notice $\delta_i \in B(F)$ and $T\delta_i \in Z$, so the norm here is the norm in Z). If $f \in B(F)$ and $\|f\|_\infty = 1$, then $f = \sum_{i=1}^{N} f(i)\delta_i$ and

$$\begin{aligned} \|Tf\| &= \left\| \sum_{i=1}^{N} T(f(i)\delta_i) \right\| \leq \sum_{i=1}^{N} \|T(f(i)\delta_i)\| \text{ by the Triangle Inequality} \\ &= \sum_{i=1}^{N} \|f(i)T(\delta_i)\| \text{ since } T \text{ is linear} \\ &= \sum_{i=1}^{N} |f(i)|\|T(\delta_i)\| \text{ by the Scalar Property} \end{aligned}$$

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Proposition 2.28 (continued)

(Continued).

$$\begin{aligned}
\|Tf\| &= \sum_{i=1}^{N} |f(i)| \|T(\delta_i)\| \text{ by the Scalar Property} \\
&\leq \sum_{i=1}^{N} 1 \|T(\delta_i)\| \text{ since } \|f\|_\infty = 1 \\
&\leq \sum_{i=1}^{N} K \text{ by the definition of } K \\
&= KN.
\end{aligned}$$

Since N and K are fixed, then $\|T\| \leq KN$ and T is bounded, as claimed. □

Theorem 2.29

Theorem 2.29. Closed and bounded subsets of $B(F)$ are compact.

Proof. Let $A \subseteq B(F)$ be closed and bounded and let (f_n) be a sequence in A. We construct a subsequence of (f_n) which converges. For each i (think of i as a position in an N-tuple) we have $(f_n(i))_{n=1}^{\infty}$ is a bounded sequence (of the ith position terms) since A is bounded. Since a bounded sequence in $\mathbb{F}$ (the scalar field; taken to be $\mathbb{R}$ or $\mathbb{C}$) has a convergent subsequence (this follows from Weierstrass's Theorem; see Theorem 2.14 in my Analysis 1 [MATH 4217/5217] notes on Section 2.3. Bolzano-Weierstrass Theorem), then there is a subsequence $(f_{n_1}(1))_{n_1=1}^{\infty}$ of $(f_n(1))_{n=1}^{\infty}$ which is convergent. Denote the limit as $f(1)$.

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Theorem 2.29 (continued)

Theorem 2.29. Closed and bounded subsets of $B(F)$ are compact.

Proof (continued). Since $F = \{1, 2, \ldots, N\}$ is a finite set, this convergence for each i implies uniform convergence over set F. But uniform convergence is equivalent to sup norm convergence by Note 2.3.A, so sequence $(f_{n_N}(i))_{n_N=1}^{\infty}$ converges to $f(i)$ in $B(F)$ (with respect to the sup norm). Now, since A is closed, it must be that

$$(f(i))_{i=1}^{N} = (f(1), f(2), \ldots, f(N)) \in A.$$

So set A is "sequentially compact" and so (by the definition of compact) set A is compact, as claimed. □

Proposition 2.30

Proposition 2.30. If T is a bijective linear operator from the normed linear space X to $B(F)$, then T is bounded.

Proof. ASSUME T is unbounded. Then from the definition of the operator norm, for each $n \in \mathbb{N}$ there exists a unit vector $z_n \in X$ such that $\|Tz_n\| \geq n$. For each such z_n define $x_n = z_n/\|Tz_n\|$.

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$$\|x_n\| = \|z_n\|/\|Tz_n\| = 1/\|Tz_n\| \leq 1/n \text{ for all } n \in \mathbb{N}.$$

But $T(x_n) = T(z_n/\|Tz_n\|) = T(z_n)/\|Tz_n\|$ is a unit vector for each $n \in \mathbb{N}$. The set of all unit vectors in $B(F)$ has as its complement two open sets: $B(0;1)$ (the open ball centered at 0 with radius 1) and $\overline{B}(0;1)^c$.

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Proposition 2.30 (continued)

Proposition 2.30. If T is a bijective linear operator from the normed linear space X to $B(F)$, then T is bounded.

Proof (continued). Since $T : X \to B(F)$ is bijective (one to one and onto) and T is linear, then there exists $T^{-1} : B(F) \to X$ that is bijective and linear. That is, $T^{-1} \in \mathcal{L}(B(F), X)$. So by Proposition 2.28, T^{-1} is bounded. Hence, by Theorem 2.6, T^{-1} is uniformly continuous on $B(F)$ and

$$T^{-1}y = T^{-1}(\lim Tx_{n_k}) = \lim(T^{-1}Tx_{n_k}) = \lim x_{n_k} = 0.$$

But y is a unit vector and so $y \neq 0$, a CONTRADICTION to the original assumption of unboundedness of T. Therefore, T is bounded. □

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Theorem 2.31

Theorem 2.31.

(a) Any linear operator $T : X \to Z$, where X is finite dimensional, is bounded.

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

(c) Any finite-dimensional subspace of a normed linear space is closed.

Theorem 2.31 (continued 1)

Theorem 2.31. (a) Any linear operator $T : X \to Z$, where X is finite dimensional, is bounded.

Proof. (a) Suppose $\{e_1, e_2, \ldots, e_n\}$ is a basis for X. Since we know some properties of linear operators to and from $B(F)$ by Propositions 2.28 and 2.30, we introduce $J : X \to B(F)$ as $Je_i = \delta_i$ where δ_i is the ith standard basis element of $B(F)$. Then J is a linear bijection from X to $B(F)$ (recall from Linear Algebra that a bijection from the basis of one vector space to the basis of another vector space is in fact a bijection between the vector spaces themselves—this is how the proof of the Fundamental Theorem of Finite Dimensional Vector Spaces goes; see Theorem 3.3.A in my online Linear Algebra [MATH 2010] notes on Section 3.3. Coordinatization of Vectors).

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Theorem 2.31 (continued 1)

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Theorem 2.31 (continued 2)

Theorem 2.31. (b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof (continued). (b) Consider $\|\cdot\|_1$ and $\|\cdot\|_2$ on finite dimensional normed linear space X. Define $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ as the identity map. By part (a), T is bounded. Let $(x_n) \to x$ with respect to $\|\cdot\|_1$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|x_n - x\|_1 < \varepsilon/\|T\|$. But then for $n \geq N$ we have, by Note 2.4.A,

$$\|x_n - x\|_2 = \|Tx_n - Tx\|_2 = \|T(x_n - x)\|_2 \leq \|T\|\|x_n - x\|_1 < \varepsilon$$

and so $(x_n) \to x$ with respect to $\|\cdot\|_2$. So $\|\cdot\|_2$ is weaker than $\|\cdot\|_1$. Similarly, taking $T : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ as the identity we can show that $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$. So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

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Theorem 2.31 (continued 3)

Theorem 2.31. (b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof (continued). For completeness, (remember these are all normed linear spaces with scalars from a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$), we know that $B(F)$ is complete under $\|\cdot\|_{\text{sup}} = \|\cdot\|_\infty$ by Theorem 2.14. Let $(X, \|\cdot\|)$ be a finite dimensional space and let (x_n) be Cauchy in $(X, \|\cdot\|)$. Define $J : X \to B(F)$ as in part (a). Then J is bijective and bounded by part (a) and so uniformly continuous by Theorem 2.6. So (Jx_n) is a Cauchy sequence in $B(F)$ since (by Note 2.4.A)

$$\|Jx_m - Jx_n\|_\infty = \|J(x_m - x_n)\|_\infty \leq \|J\|\|x_m - x_n\|$$

and so $(Jx_n) \to y$ for some $y \in B(F)$ since $B(F)$ is complete.

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Theorem 2.31. (b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof (continued). Since J is bijective and linear then $J^{-1} : B(F) \to X$ is bijective and linear and so bounded by Proposition 2.30. So

$$\|x_n - J^{-1}y\| = \|J^{-1}Jx_n - J^{-1}y\| = \|J^{-1}(Jx_n - y)\| \leq \|J^{-1}\|\|Jx_n - y\|_\infty$$

and since $\|Jx_n - y\|_\infty \to 0$ and J^{-1} is bounded, then $(x_n) \to J^{-1}y$ and (x_n) converges. Since (x_n) is an arbitrary Cauchy sequence, then $(X, \|\cdot\|)$ is complete, as claimed. □

Theorem 2.31 (continued 5).

Theorem 2.31. (c) Any finite-dimensional subspace of a normed linear space is closed.

Proof (continued). (c) Let $(X, \|\cdot\|)$ be a finite dimensional subspace of a given space. By part (b) the space $(X, \|\cdot\|)$ is complete and so is a Banach space. Since $(X, \|\cdot\|)$ is a subspace of itself, then by Theorem 2.16, $(X, \|\cdot\|)$ is closed. □

Theorem 2.32

Theorem 2.32. A linear operator $T : X \to Y$, where Y is finite-dimensional, is bounded if and only if $N(T)$ (the nullspace of T) is closed.

Proof. Suppose T is bounded. Then T is continuous by Proposition 2.6. Now $N(T)$ is the inverse image of the closed set $\{0\}$. Continuous mappings have inverse images of closed sets closed (as seen in Analysis 1 [MATH 4217/5217]; see my online notes on Section 4.1. Limits and Continuity), so $N(T)$ is closed. Notice that we did not use the finite-dimensional hypothesis here.

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Suppose $N(T)$ is closed. Consider the quotient space $X/N(T)$. The mapping $\tilde{T} : X/N(T) \to Y$ defined $\tilde{T}\overline{x} = Tx$ is one to one (injective); see Note 2.7.A. Since Y is hypothesized to be finite dimensional, then $X/N(T)$ must be finite dimensional (an infinite dimensional space cannot be mapped injectively to a finite dimensional space; consider how the basis is mapped). So by Theorem 2.31(a), $\tilde{T}$ is bounded.

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Theorem 2.32 (continued)

Theorem 2.32. A linear operator $T : X \to Y$, where Y is finite-dimensional, is bounded if and only if $N(T)$ (the nullspace of T) is closed.

Proof (continued). Now $T = \tilde{T}\pi_{N(T)}$, where $\pi_{N(T)} : X \to X/N(T)$ is the canonical projection with respect to $N(T)$ and $x \mapsto \overline{x}$. Since $\tilde{T}$ is bounded and $\|\pi_{N(T)}\| = 1$ by Theorem 2.27(c) then, by Proposition 2.8,

$$\|T\| = \|\tilde{T}\pi_{N(T)}\| \leq \|\tilde{T}\|\|\pi_{N(T)}\| = \|\tilde{T}\|.$$

So T is bounded, as claimed. □

Theorem 2.33

Theorem 2.33. Riesz's Lemma.
Given a closed, proper subspace M of a normed linear space $(X, \|\cdot\|)$ and given $\varepsilon > 0$, there is a unit vector $x \in X$ such that $d(x, M) \geq 1 - \varepsilon$.

Proof. Let $\varepsilon > 0$ (and less than 1). Let $y \in X$, $y \notin M$, and define $r = d(y, M)$. Since M is closed then $r > 0$. Since $d(y, M) = \inf\{\|y - m\| \mid m \in M\}$, then there is $z \in M$ such that $\|y - z\| \leq r/(1 - \varepsilon)$ since $r/(1 - \varepsilon) > r$. For any $v \in M$, we have that $z + v \in M$ since M is a subspace. So $\|y - (z + v)\| \geq r$ and so $d(y - z, M) \geq r \geq (1 - \varepsilon)\|y - z\|$ (by the restriction $\|y - z\| \leq r/(1 - \varepsilon)$ from above).

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Given a closed, proper subspace M of a normed linear space $(X, \|\cdot\|)$ and given $\varepsilon > 0$, there is a unit vector $x \in X$ such that $d(x, M) \geq 1 - \varepsilon$.

Proof. Let $\varepsilon > 0$ (and less than 1). Let $y \in X$, $y \notin M$, and define $r = d(y, M)$. Since M is closed then $r > 0$. Since $d(y, M) = \inf\{\|y - m\| \mid m \in M\}$, then there is $z \in M$ such that $\|y - z\| \leq r/(1 - \varepsilon)$ since $r/(1 - \varepsilon) > r$. For any $v \in M$, we have that $z + v \in M$ since M is a subspace. So $\|y - (z + v)\| \geq r$ and so $d(y - z, M) \geq r \geq (1 - \varepsilon)\|y - z\|$ (by the restriction $\|y - z\| \leq r/(1 - \varepsilon)$ from above). Define $x = (y - z)/\|y - z\|$. Then $x \in X$ is a unit vector and

$$\begin{aligned} d(x, M) &= d((y - z)/\|y - z\|, M) = (1/\|y - z\|)d(y - z, M) \\ &\quad \text{since } d(\alpha w, M) = |\alpha| d(w, M) \text{ for all } w \in X \\ &\geq \left(\frac{1}{\|y - z\|}\right)((1 - \varepsilon)\|y - z\|) = 1 - \varepsilon. \end{aligned}$$

So x a vector with the desired properties. □

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Theorem 2.34. Reisz's Theorem

Theorem 2.34. Riesz's Theorem. A normed linear space $(X, \|\cdot\|)$ is finite-dimensional if and only if the closed unit ball $\overline{B}(0;1)$ is compact.

Proof. First, suppose that X is infinite-dimensional. We create a sequence of unit vectors, (x_n), as follows. Let x_1 be any unit vector in X. With $\{x_1, x_2, \ldots, x_n\}$ chosen, define $M_n = \text{span}\{x_1, x_2, \ldots, x_n\}$. Then M_n is a finite dimensional subspace of X, and so it is closed (by Theorem 2.31(c)) and so not equal to X (since X is infinite dimensional).

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Theorem 2.34. Riesz's Theorem (continued)

Theorem 2.34. Riesz's Theorem. A normed linear space $(X, \|\cdot\|)$ is finite-dimensional if and only if the closed unit ball $\overline{B}(0;1)$ is compact.

Proof (continued). Suppose X is finite dimensional. Then by Theorem 2.31(b), all norms on X are equivalent (and so the properties of closed and boundedness are the same with respect to any norm on X), so we can assume the norm is the sup norm. By the Fundamental Theorem of Finite Dimensional Vector Spaces (see Theorem 3.3.A of my online Linear Algebra [MATH 2010] notes on Section 3.3. Coordinatization of Vectors), X and $B(F)$ are isomorphic (where $F = \{1, 2, \ldots, N\}$ and N is the dimension of X). Since $\overline{B}(0;1)$ is closed and bounded then, by Theorem 2.29, $\overline{B}(0;1)$ is compact in $B(F)$ and hence in X. □