Chapter 2. Normed Linear Spaces: The Basics

2.8. Finite Dimensional Normed Linear Spaces—Proofs of Theorems
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Proposition 2.28

For any normed linear space $Z$, all elements of $\mathcal{L}(B(F), Z)$ (the set of linear operators from $B(F)$ to $Z$) are bounded.

Proof. For any $T \in \mathcal{L}(B(F), Z)$, let $K = \max\{\|T\delta_i\|\}$ (notice $\delta_i \in F$ and $T\delta_i \in Z$, so the norm here is the norm in $Z$).
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$$
\|Tf\| = \left\| \sum_{i=1}^{N} T(f(i)\delta_i) \right\| \leq \sum_{i=1}^{N} \|T(f(i)\delta_i)\| \text{ by the Triangle Inequality}
$$

$$
= \sum_{i=1}^{N} \|f(i)T(\delta_i)\| \text{ since } T \text{ is linear}
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= \sum_{i=1}^{N} |f(i)|\|T(\delta_i)\| \text{ by the Scalar Property}
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(Continued).

\[ \| Tf \| = \sum_{i=1}^{N} |f(i)| \| T(\delta_i) \| \text{ by the Scalar Property} \]

\[ \leq \sum_{i=1}^{N} 1 \| T(\delta_i) \| \text{ since } \| f \| = 1 \]

\[ \leq \sum_{i=1}^{N} K \text{ by the definition of } K \]

\[ = KN. \]

Since \( N \) and \( K \) are fixed, \( T \) is bounded.
Theorem 2.29. Closed and bounded subsets of $B(F)$ are compact.

Proof. Let $A \subseteq B(f)$ be closed and bounded and let $(f_n)$ be a sequence in $A$. We construct a subsequence of $(f_n)$ which converges. For each $i$ (think of $i$ as a position in an $n$-tuple) we have $\|f_n(i)\| \leq \|f_n\|_{\text{sup}}$, so $(f_n(i))_{n=1}^{\infty}$ is a bounded sequence (of the $i$th position terms).
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Theorem 2.29 (continued)

**Theorem 2.29.** Closed and bounded subsets of $B(F)$ are compact.

**Proof (continued).** Since $F = \{1, 2, \ldots, N\}$ is a finite set, this convergence for each $i$ implies uniform convergence over set $F$. But uniform convergence is equivalent to sup norm convergence (see page 19), so sequence $f_{n_N}(i))_{n=1}^{\infty}$ converges to $f(i)$ in $B(F)$ (with respect to the sup norm).
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$$f(i)_{i=1}^N = (f(1), f(2), \ldots, f(N)) \in A.$$  

So, by the second form of the definition of compact (see page 17), set $A$ is compact.
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Proposition 2.30

Proposition 2.30. If $T$ is a bijective, linear operator from the normed linear space $X$ to $B(F)$, then $T$ is bounded.

Proof. ASSUME not; suppose $T$ is not bounded. Then from the definition of the operator norm, for each $n \in \mathbb{N}$ there exists a unit vector $z_n \in X$ such that $\|Tz_n\| \geq n$. For each such $z_n$ define $x_n = x_n/\|Tz_n\|$.
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$$\| x_n \| = \| z_n / \| Tz_n \| \| = 1 / \| Tz_n \| \leq 1 / n$$

for all $n \in \mathbb{N}$.

But $T(x_n) = T(z_n / \| Tz_n \|) = T(z_n) / \| Tz_n \|$ is a unit vector for each $n \in \mathbb{N}$. The set of all unit vectors in $B(F)$ has as its complement two open sets: $B(0; 1)$ (the open ball centered at 0 with radius 1) and $\overline{B}(0; 1)^c$. 
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Proposition 2.30. If $T$ is a bijective, linear operator from the normed linear space $X$ to $B(F)$, then $T$ is bounded.

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But $T(x_n) = T(z_n/\|Tz_n\|) = T(z_n)/\|Tz_n\|$ is a unit vector for each $n \in \mathbb{N}$. The set of all unit vectors in $B(F)$ has as its complement two open sets: $B(0; 1)$ (the open ball centered at 0 with radius 1) and $\overline{B}(0; 1)^c$. So the complement is open and the set of all unit vectors in $B(F)$ is closed (and, of course, bounded). By Proposition 2.29, this set is compact. So, by the second definition of compact, there is a subsequence of $(Tx_n)$, say $(Tx_{n_k})$, which converges to some $y$ in the set of unit vectors in $B(F)$. 
Proposition 2.30. If $T$ is a bijective, linear operator from the normed linear space $X$ to $B(F)$, then $T$ is bounded.

Proof (continued). Since $T : X \rightarrow B(F)$ is bijective (one to one and onto) and $T$ is linear, then there exists $T^{-1} : B(F) \rightarrow X$ that is bijective and linear. That is, $T^{-1} \in \mathcal{L}(B(F), X)$. So by Proposition 2.28, $T^{-1}$ is bounded. Hence, by Theorem 2.6, $T^{-1}$ is uniformly continuous on $B(F)$ and

$$T^{-1}y = T^{-1}\left(\lim Tx_{n_k}\right) = \lim\left(T^{-1}Tx_{n_k}\right) = \lim x_{n_k} = 0.$$ But $y$ is a unit vector and so $y \neq 0$, a CONTRADICTION to the original assumption of unboundedness of $T$. Therefore, $T$ is bounded.
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$$T^{-1}y = T^{-1}(\lim Tx_{n_k}) = \lim(T^{-1}Tx_{n_k}) = \lim x_{n_k} = 0.$$  

But $y$ is a unit vector and so $y \neq 0$, a CONTRADICTION to the original assumption of unboundedness of $T$. Therefore, $T$ is bounded. \qed
Theorem 2.31.

(a) Any linear operator $T : X \to Z$, where $X$ is finite dimensional, is bounded.

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

(c) Any finite-dimensional subspace of a normed linear space is closed.
Theorem 2.31(a)

(a) Any linear operator $T : X \rightarrow Z$, where $X$ is finite dimensional, is bounded.

Proof of (a). Suppose \( \{ e_1, e_2, \ldots, e_n \} \) is a basis for $X$. Since we know some properties of linear operators to and from $B(F)$ by Propositions 2.28 and 2.30, we introduce $J : X \rightarrow B(F)$ as $Je_i = \delta_i$ where $\delta_i$ is the $i$th standard basis element of $B(F)$. 
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(a) Any linear operator $T : X \to Z$, where $X$ is finite dimensional, is bounded.

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Theorem 2.31(a)

(a) Any linear operator \( T : X \to Z \), where \( X \) is finite dimensional, is bounded.

Proof of (a). Suppose \( \{e_1, e_2, \ldots, e_n\} \) is a basis for \( X \). Since we know some properties of linear operators to and from \( B(F) \) by Propositions 2.28 and 2.30, we introduce \( J : X \to B(F) \) as \( Je_i = \delta_i \) where \( \delta_i \) is the \( i \)th standard basis element of \( B(F) \). Then \( J \) is a linear bijection from \( X \) to \( B(F) \) (recall from Linear Algebra that a bijection from the basis of one vector space to the basis of another vector space is in fact a bijection between the vector spaces themselves—this is how the proof of the Fundamental Theorem of Finite Dimensional Vector Spaces goes). So \( J^{-1} : B(F) \to X \) exists (and is linear) and we have \( TJ^{-1} : B(F) \to Z \) is linear (a composition of linear operators is linear) and so by Proposition 2.28, \( TJ^{-1} \) is bounded. By Proposition 2.30, \( J \) is bounded. Next, \( T = (TJ^{-1})J \) and so by Proposition 2.8, \( \|T\| \leq \|TJ^{-1}\|||J|| \) and so \( T \) is bounded.
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(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $F$ (where $F$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

**Proof of (b).** Consider $\| \cdot \|_1$ and $\| \cdot \|_2$ on finite dimensional normed linear space $X$. Define $T : (X, \| \cdot \|_1) \to (X, \| \cdot \|_2)$ as the identity map. By part (a), $T$ is bounded. Let $(x_n) \to x$ with respect to $\| \cdot \|_1$.
Theorem 2.31(b)

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$$\|x_n - x\|_2 = \|Tx_n - Tx\|_2 = \|T(x_n - x)\|_2 \leq \|T\| \|x_n - x\|_1 \leq \varepsilon$$

and so $(x_n) \rightarrow x$ with respect to $\| \cdot \|_2$. So $\| \cdot \|_2$ is weaker than $\| \cdot \|_1$. 
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and so $(x_n) \rightarrow x$ with respect to $\| \cdot \|_2$. So $\| \cdot \|_2$ is weaker than $\| \cdot \|_1$. Similarly, taking $T : (X, \| \cdot \|_2) \rightarrow (X, \| \cdot \|_1)$ as the identity we can show that $\| \cdot \|_1$ is weaker than $\| \cdot \|_2$. So $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.
(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field \( F \) (where \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \)) are complete.

**Proof of (b).** Consider \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) on finite dimensional normed linear space \( X \). Define \( T : (X, \| \cdot \|_1) \to (X, \| \cdot \|_2) \) as the identity map. By part (a), \( T \) is bounded. Let \( (x_n) \to x \) with respect to \( \| \cdot \|_1 \). Then for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have \( \| x_n - x \|_1 < \varepsilon / \| T \| \). But then for \( n \geq N \) we have

\[
\| x_n - x \|_2 = \| Tx_n - Tx \|_2 = \| T(x_n - x) \|_2 \leq \| T \| \| x_n - x \|_1 \leq \varepsilon
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and so \( (x_n) \to x \) with respect to \( \| \cdot \|_2 \). So \( \| \cdot \|_2 \) is weaker than \( \| \cdot \|_1 \). Similarly, taking \( T : (X, \| \cdot \|_2) \to (X, \| \cdot \|_1) \) as the identity we can show that \( \| \cdot \|_1 \) is weaker than \( \| \cdot \|_2 \). So \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent.
Theorem 2.31(b) (continued)

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $F$ (where $F$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof of (b) (continued). For completeness, (remember these are all normed linear spaces with scalars from a field $F$ where $F$ is either $\mathbb{R}$ or $\mathbb{C}$), we know that $B(F)$ is complete under $\| \cdot \|_{\sup} = \| \cdot \|_{\infty}$ by Theorem 2.14. Let $(X, \| \cdot \|)$ be a finite dimensional space and let $(x_n)$ be Cauchy in $(X, \| \cdot \|)$. Define $J : X \to B(F)$ as in part (a). Then $J$ is bijective and bounded by (a) and so uniformly continuous by Theorem 2.6.
(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $F$ (where $F$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof of (b) (continued). For completeness, (remember these are all normed linear spaces with scalars from a field $F$ where $F$ is either $\mathbb{R}$ or $\mathbb{C}$), we know that $B(F)$ is complete under $\| \cdot \|_{\sup} = \| \cdot \|_{\infty}$ by Theorem 2.14. Let $(X, \| \cdot \|)$ be a finite dimensional space and let $(x_n)$ be Cauchy in $(X, \| \cdot \|)$. Define $J : X \to B(F)$ as in part (a). Then $J$ is bijective and bounded by (a) and so uniformly continuous by Theorem 2.6. So $(Jx_n)$ is a Cauchy sequence in $B(F)$ since

$$\|Jx_m - Jx_n\|_{\infty} = \|J(x_m - x_n)\|_{\infty} \leq \|J\| \|x_m - x_n\|$$

and so $(Jx_n) \to y$ for some $y \in B(F)$ since $B(F)$ is complete.
Theorem 2.31(b) (continued)

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $F$ (where $F$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof of (b) (continued). For completeness, (remember these are all normed linear spaces with scalars from a field $F$ where $F$ is either $\mathbb{R}$ or $\mathbb{C}$), we know that $B(F)$ is complete under $\| \cdot \|_{\text{sup}} = \| \cdot \|_{\infty}$ by Theorem 2.14. Let $(X, \| \cdot \|)$ be a finite dimensional space and let $(x_n)$ be Cauchy in $(X, \| \cdot \|)$. Define $J : X \to B(F)$ as in part (a). Then $J$ is bijective and bounded by (a) and so uniformly continuous by Theorem 2.6. So $(Jx_n)$ is a Cauchy sequence in $B(F)$ since

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and so $(Jx_n) \to y$ for some $y \in B(F)$ since $B(F)$ is complete.
(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $F$ (where $F$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

Proof of (b) (further continued). Since $T$ is bijective and linear then $J^{-1} : B(F) \to X$ is bijective and linear and so bounded by Proposition 2.30. So

$$\|x_n - J^{-1}y\| = \|J^{-1}Jx_n - J^{-1}y\| = \|J^{-1}(Jx_n - y)\| \leq \|J^{-1}\| \|Jx_n - y\|_\infty$$

and since $\|Jx_n - y\|_\infty \to 0$ and $J^{-1}$ is bounded, then $(x_n) \to J^{-1}y$ and $(x_n)$ converges. That is, $(X, \| \cdot \|)$ is complete. \qed
(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

**Proof of (b) (further continued).** Since $T$ is bijective and linear then $J^{-1} : B(F) \rightarrow X$ is bijective and linear and so bounded by Proposition 2.30. So

$$
\|x_n - J^{-1}y\| = \|J^{-1}Jx_n - J^{-1}y\| = \|J^{-1}(Jx_n - y)\| \leq \|J^{-1}\| \|Jx_n - y\|_{\infty}
$$

and since $\|Jx_n - y\|_{\infty} \to 0$ and $J^{-1}$ is bounded, then $(x_n) \to J^{-1}y$ and $(x_n)$ converges. That is, $(X, \| \cdot \|)$ is complete. \qed
Theorem 2.31(c).

(c) Any finite-dimensional subspace of a normed linear space is closed.

Proof of (c). Let \((X, \| \cdot \|)\) be a finite dimensional subspace of a given space. By part (b) the space \((X, \| \cdot \|)\) is complete and so is a Banach space. Since \((X, \| \cdot \|)\) is a subspace of itself, then by Theorem 2.16, \((X, \| \cdot \|)\) is closed.
(c) Any finite-dimensional subspace of a normed linear space is closed.

Proof of (c). Let \((X, \| \cdot \|)\) be a finite dimensional subspace of a given space. By part (b) the space \((X, \| \cdot \|)\) is complete and so is a Banach space. Since \((X, \| \cdot \|)\) is a subspace of itself, then by Theorem 2.16, \((X, \| \cdot \|)\) is closed.
Theorem 2.32

**Theorem 2.32.** A linear operator $T : X \to Y$, where $Y$ is finite-dimensional, is bounded if and only if $N(T)$ (the nullspace of $T$) is closed.

**Proof.** Suppose $T$ is bounded. Then $T$ is continuous by Proposition 2.6. Now $N(T)$ is the inverse image of the closed set $\{0\}$. Continuous mappings have inverse images of closed sets closed (as seen in senior level analysis), so $N(T)$ is closed. Notice that we did not use the finite-dimensional hypothesis here.
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Suppose $N(T)$ is closed. Consider the quotient space $X/N(T)$. The mapping $\tilde{T} : X/N(T) \to Y$ defined $\tilde{T}x = Tx$ is one to one (injective); see section 2.7.
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Suppose $N(T)$ is closed. Consider the quotient space $X/N(T)$. The mapping $\tilde{T} : X/N(T) \to Y$ defined $\tilde{T}x = Tx$ is one to one (injective); see section 2.7. Since $Y$ is hypothesized to be finite dimensional, then $X/N(T)$ must be finite dimensional (a finite dimensional space cannot be mapped onto an infinite dimensional space). So by Theorem 2.31(a), $\tilde{T}$ is bounded.
Theorem 2.32. A linear operator $T : X \rightarrow Y$, where $Y$ is finite-dimensional, is bounded if and only if $N(T)$ (the nullspace of $T$) is closed.

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Proof (continued). Now $T = \tilde{T} \pi_{N(T)}$, where $\pi_{N(T)}$ is the canonical projection with respect to $N(T)$ (see Section 2.7). Since $\tilde{T}$ is bounded and $\pi_{N(T)} = 1$ by Theorem 2.27(c) then, by Proposition 2.8,

$$\| T \| = \| \tilde{T} \pi_{N(T)} \| \leq \| \tilde{T} \| \| \pi_{N(T)} \| = \| \tilde{T} \| .$$

So $T$ is bounded, as claimed.
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Theorem 2.33. Riesz’s Lemma.
Given a closed, proper subspace $M$ of a normed linear space $(X, \| \cdot \|)$ and given $\varepsilon > 0$, there is a unit vector $x \in X$ such that $d(x, M) \geq 1 - \varepsilon$.

Proof. Let $\varepsilon > 0$ (and less than 1). Let $y \in X$, $y \notin M$, and define $r = d(y, M)$. Since $M$ is closed, $r > 0$. Define $x = \frac{y}{\|y - z\|}$. Then $x$ is a unit vector and $d(x, M) = d(y - z, M) / \|y - z\| = \varepsilon$. So $x$ is a vector with the desired properties.
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Proof. Let $\varepsilon > 0$ (and less than 1). Let $y \in X$, $y \notin M$, and define $r = d(y, M)$. Since $M$ is closed, $r > 0$. Since $d(y, M) = \inf \{ \| y - m \| \mid m \in M \}$, then there is $z \in M$ such that $\| y - z \| \leq r/(1 - \varepsilon)$ since $r/(1 - \varepsilon) > r$. For any $v \in M$, we have that $z + v \in M$ since $M$ is a subspace. So $\| y - (z + v) \| \geq r$ and so $d(y - z, M) \geq r \geq (1 - \varepsilon)\| y - z \|$ (by the restriction $\| y - z \| \leq r/(1 - \varepsilon)$ from above).
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**Proof.** Let $\varepsilon > 0$ (and less than 1). Let $y \in X$, $y \notin M$, and define $r = d(y, M)$. Since $M$ is closed, $r > 0$. Since $d(y, M) = \inf\{\|y - m\| \mid m \in M\}$, then there is $z \in M$ such that $\|y - z\| \leq r/(1 - \varepsilon)$ since $r/(1 - \varepsilon) > r$. For any $v \in M$, we have that $z + v \in M$ since $M$ is a subspace. So $\|y - (z + v)\| \geq r$ and so $d(y - z, M) \geq r \geq (1 - \varepsilon)\|y - z\|$ (by the restriction $\|y - z\| \leq r/(1 - \varepsilon)$ from above). Define $x = (y - z)/\|y - z\|$. Then $x \in X$ is a unit vector and

$$d(x, M) = d((y - z)/\|y - z\|, M) = d(y - z, M)/\|y - z\|$$

since $d(\alpha w, M) = \alpha d(w, M)$ for all $w \in X$

$$\geq \frac{1}{\|y - z\|}(1 - \varepsilon)\|y - z\| = 1 - \varepsilon.$$ 

So $x$ a vector with the desired properties.
Theorem 2.33

Theorem 2.33. Riesz’s Lemma.

Given a closed, proper subspace \( M \) of a normed linear space \((X, \| \cdot \|)\) and given \( \varepsilon > 0 \), there is a unit vector \( x \in X \) such that \( d(x, M) \geq 1 - \varepsilon \).

Proof. Let \( \varepsilon > 0 \) (and less than 1). Let \( y \in X, y \not\in M \), and define \( r = d(y, M) \). Since \( M \) is closed, \( r > 0 \). Since \( d(y, M) = \inf \{ \| y - m \| \mid m \in M \} \), then there is \( z \in M \) such that \( \| y - z \| \leq r/(1 - \varepsilon) \) since \( r/(1 - \varepsilon) > r \). For any \( v \in M \), we have that \( z + v \in M \) since \( M \) is a subspace. So \( \| y - (z + v) \| \geq r \) and so \( d(y - z, M) \geq r \geq (1 - \varepsilon) \| y - z \| \) (by the restriction \( \| y - z \| \leq r/(1 - \varepsilon) \) from above). Define \( x = (y - z)/\| y - z \| \). Then \( x \in X \) is a unit vector and

\[
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since \( d(\alpha w, M) = \alpha d(w, M) \) for all \( w \in X \)

\[
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\]

So \( x \) a vector with the desired properties.
Theorem 2.34. Reisz’s Theorem

Theorem 2.34. Reisz’s Theorem. A normed linear space \((X, \| \cdot \|)\) is finite-dimensional if and only if the closed unit ball \(\overline{B}(0; 1)\) is compact.

Proof. First, suppose that \(X\) is infinite-dimensional. We create a sequence of unit vectors, \((x_n)\), as follows. Let \(x_1\) be any unit vector in \(X\). With \(\{x_1, x_2, \ldots, x_n\}\) chosen, define \(M_n = \text{span}\{x_1, x_2, \ldots, x_n\}\).
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Theorem 2.34. Riesz’s Theorem. A normed linear space \((X, \| \cdot \|)\) is finite-dimensional if and only if the closed unit ball \(\overline{B}(0; 1)\) is compact.

Proof (continued). Second, \(\overline{B}(1)\) is closed and bounded and so in finite dimensional \((X, \| \cdot \|)\) we have that \(\overline{B}(1)\) is compact by the Heine Borel Theorem. (The text gives a “proof” based on Theorems 2.29 and 2.31(b)).