Chapter 3. Major Banach Space Theorems

3.2. Baire Category Theorem—Proofs of Theorems

Proposition 3.1. The Nested Set Theorem.
Given a sequence \( F_1 \supseteq F_2 \supseteq F_3 \cdots \) of closed nonempty sets in a Banach space such that \( \text{diam}(F_n) \to 0 \), there is a unique point that is in \( F_n \) for all \( n \in \mathbb{N} \).

**Proof.** Choose some \( x_n \in F_n \) for each \( n \in \mathbb{N} \). Since \( \text{diam}(F_n) \to 0 \), for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( n \geq N \) then \( \text{diam}(F_n) < \varepsilon \). Since the \( F_n \) are nested, then for all \( m, n \geq N \), we have \( \|x_n - x_m\| < \varepsilon \), and so \( (x_n) \) is a Cauchy sequence. Since we are in a Banach space, there is \( x \) such that \( (x_n) \to x \). Now for \( n \in \mathbb{N} \), the sequence \( (x_n, x_{n+1}, x_{n+2}, \ldots) \subseteq F_n \) is convergent to \( x \) and since \( F_n \) is closed, then \( x \in F_n \). That is \( x \in F_n \) for all \( n \in \mathbb{N} \).

Next, suppose both \( x, y \in F_n \) for all \( n \in \mathbb{N} \). Then \( \|x - y\| \leq \text{diam}(F_n) \) for all \( n \in \mathbb{N} \) and since \( \text{diam}(F_n) \to 0 \), then \( \|x - y\| = 0 \), or \( x = y \), establishing uniqueness.

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Proposition 3.2, Baire’s Theorem.
The intersection of countably many open and dense sets in a Banach space is dense.

**Proof.** Let \( (V_n) \) be a sequence of dense open subsets of a given Banach space. Let \( W \) be any nonempty open set. We show \( \bigcap_{n \in \mathbb{N}} V_n \) is dense by finding \( x \in \bigcap_{n \in \mathbb{N}} V_n \) such that \( x \in W \). Now \( W \cap V_1 \) is open and nonempty (since \( V_1 \) is dense in the space). So there is some closed ball \( \overline{B}_1 \) that is a subset of \( W \cap V_1 \) where the radius of \( \overline{B}_1 \) is less than 1. Inductively we produce closed balls \( \overline{B}_n \) such that \( \overline{B}_n \) is a subset of \( \overline{B}_{n-1} \cap V_n \) and the radius of \( \overline{B}_n \) is less than \( 1/n \). Notice that \( \overline{B}_n \subseteq \overline{B}_{n-1} \) by construction and so the sequence \( (\overline{B}_n) \) is a sequence of nested closed sets and \( \text{diam}(\overline{B}_n) \to 0 \). So by the Nested Set Theorem (Proposition 3.1) there is a unique \( x \in \overline{B}_n \) for all \( n \in \mathbb{N} \). So \( x \in W \) and \( x \in V_n \) for all \( n \in \mathbb{N} \).

Therefore \( x \in W \) and \( x \in \bigcap_{n \in \mathbb{N}} V_n \) and therefore \( \bigcap_{n \in \mathbb{N}} V_n \) is dense in the space.

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Corollary 3.3, Dual Form of Baire’s Theorem.
In any Banach space, a countable union of closed sets with empty interiors has an empty interior.

**Proof.** Let \( X_n \) be closed sets with empty interiors in Banach space \( (X, \| \cdot \|) \). Then \( X \setminus X_n \) is dense in \( X \) for each \( n \in \mathbb{N} \). So by Baire’s Theorem (Theorem 3.2) we have that

\[
\bigcap_{n \in \mathbb{N}} (X \setminus X_n) = X \setminus \bigcup_{n \in \mathbb{N}} X_n \quad \text{(by DeMorgan’s Laws)}
\]

is dense in \( X \) and so has an empty interior.