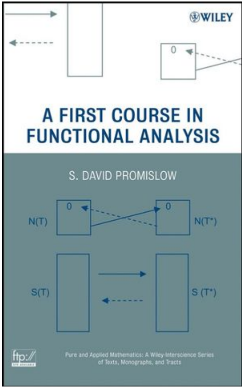
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems

3.2. Baire Category Theorem—Proofs of Theorems



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Given a sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ of closed nonempty sets in a Banach space such that $\text{diam}(F_n) \to 0$, there is a unique point that is in F_n for all $n \in \mathbb{N}$.

Proof. Choose some $x_n \in F_n$ for each $n \in \mathbb{N}$. Since $\text{diam}(F_n) \to 0$, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\text{diam}(F_n) < \varepsilon$. Since the F_n are nested, then for all $m, n \geq N$, we have $\|x_n - x_m\| < \varepsilon$, and so (x_n) is a Cauchy sequence. Since we are in a Banach space, there is x such that $(x_n) \to x$. Now for $n \in \mathbb{N}$, the sequence $(x_n, x_{n+1}, x_{n+2}, \ldots) \subseteq F_n$ is convergent to x and since F_n is closed then $F_n = \overline{F}_n$. So $x \in F_n$ by Theorem 2.2.A(iii). That is $x \in F_n$ for all $n \in \mathbb{N}$.

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Next, suppose both $x, y \in F_n$ for all $n \in \mathbb{N}$. Then $\|x - y\| \leq \text{diam}(F_n)$ for all $n \in \mathbb{N}$ and since $\text{diam}(F_n) \to 0$, then $\|x - y\| = 0$, or $x = y$, establishing uniqueness. □

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Proposition 3.2. Baire's Theorem

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The intersection of countably many open and dense sets in a Banach space is dense.

Proof. Let (V_n) be a sequence of dense open subsets of a given Banach space. Let W be any nonempty open set in the Banach space. We show $\cap_{n\in\mathbb{N}} V_n$ is dense in the Banach space by finding $x \in \cap_{n\in\mathbb{N}} V_n$ such that $x \in W$ (so that $\cap_{n\in\mathbb{N}} V_n$ intersects every open subset of the Banach space and hence by Note 3.2.A, $\cap_{n\in\mathbb{N}} V_n$ is dense in the Banach space).

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Proposition 3.2. Baire's Theorem (continued)

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Proof (continued). Notice that $\overline{B}_n \subseteq \overline{B}_{n-1}$ by construction and so the sequence $(\overline{B}_n)$ is a sequence of nested closed sets and $\mathrm{diam}(\overline{B}_n) \to 0$. So by the Nested Set Theorem (Proposition 3.1) there is a unique $x \in \overline{B}_n$ for all $n \in \mathbb{N}$. So $x \in W$ and $x \in V_n$ for all $n \in \mathbb{N}$ (since $\overline{B}_n \subseteq W$ and $\overline{B}_n \subseteq V_n$ for all $n \in \mathbb{N}$). Therefore $x \in W$ and $x \in \cap V_n$ and so $\cap V_n$ is dense in the space (by Note 3.2.A). $\square$

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In any Banach space, a countable union of closed sets with empty interiors has an empty interior.

Proof. Let X_n be closed sets with empty interiors in Banach space X. Then $X \setminus X_n$ is dense in X for each $n \in \mathbb{N}$ by Note 3.2.B. So by Baire's Theorem (Theorem 3.2) we have that

$$\cap_{n\in\mathbb{N}}(X \setminus X_n) = X \setminus (\cup_{n\in\mathbb{N}} X_n) \text{ (by DeMorgan's Laws)}$$

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