Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems
3.2. Baire Category Theorem—Proofs of Theorems
Proposition 3.1. The Nested Set Theorem.
Given a sequence $F_1 \supseteq F_2 \supseteq F_3 \cdots$ of closed nonempty sets in a Banach space such that $\text{diam}(F_n) \to 0$, there is a unique point that is in $F_n$ for all $n \in \mathbb{N}$.

Proof. Choose some $x_n \in F_n$ for each $n \in \mathbb{N}$. 
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The intersection of countably many open and dense sets in a Banach space is dense.

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Corollary 3.3. Dual Form of Baire’s Theorem.
In any Banach space, a countable union of closed sets with empty interiors has an empty interior.

Proof. Let \( X_n \) be closed sets with empty interiors in Banach space \( (X, \| \cdot \|) \).
Corollary 3.3. Dual Form of Baire’s Theorem.
In any Banach space, a countable union of closed sets with empty interiors has an empty interior.

Proof. Let $X_n$ be closed sets with empty interiors in Banach space $(X, \| \cdot \|)$. Then $X \setminus X_n$ is dense in $X$ for each $n \in \mathbb{N}$. 
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$$\bigcap_{n \in \mathbb{N}} (X \setminus X_n) = X \setminus (\bigcup_{n \in \mathbb{N}} X_n)$$

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