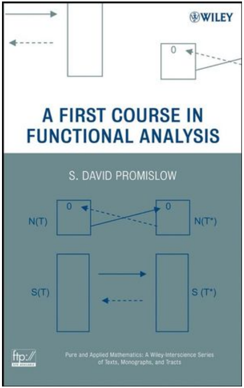
Introduction to Functional Analysis

Chapter 3. Major Banach Space Theorems

3.3. Open Mappings—Proofs of Theorems



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Theorem 3.5. Open Mapping Theorem.
Given a surjective (onto) $T \in \mathcal{B}(X, Y)$ where X and Y are Banach spaces, if $U \subseteq X$ is open then $T(U)$ is open.

Proof. We follow the steps given in the text.

STEP 1. Suppose that we have shown that $T(B(1)) \supseteq B'(\delta)$ for some $\delta > 0$. Then given any open $U \subseteq X$ and $y \in T(U)$, we have $y = Tx$ for some $x \in U$. Since U is open, there is $r > 0$ such that $U \supseteq B(x;r) = x + rB(0;1) = x + rB(1)$, by Lemma 3.4(a). Therefore

$$\begin{aligned} T(U) &\supseteq T(x + rB(1)) = Tx + rT(B(1)) \text{ since } T \text{ is linear} \\ &= y + rT(B(1)) \supseteq y + rB'(\delta) = y + B'(r\delta) \text{ by Lemma 3.4(a)} \\ &= B'(y; r\delta). \end{aligned}$$

Therefore $T(U)$ is open.

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Theorem 3.5 (continued 1)

Theorem 3.5. Open Mapping Theorem.
Given a surjective (onto) $T \in \mathcal{B}(X, Y)$ where X and Y are Banach spaces, if $U \subseteq X$ is open then $T(U)$ is open.

Proof (continued). STEP 2. Let $V = T(B(0; 1/2)) = T(B(1/2))$. Notice that $V = -V$. For any $y \in Y$, since T is onto, there is $x \in X$ such that $y = Tx = 2kT(x/(2k))$. So if $\|x\| < k$ then $\|x/(2k)\| < 1/2$ and $x/(2k) \in B(1/2)$, so that $y \in 2kT(B(1/2)) = 2kV$. So $y \in nV$ for some $n \in \mathbb{N}$ and therefore $Y = \cup_{n\in\mathbb{N}} nV$. Now consider the sequence of closed sets $(\overline{nV})_{n=1}^{\infty}$. If each $\overline{nV} = n\overline{V}$ has an empty interior, then (by Corollary 3.3, the Dual Form of Baire's Theorem) $\cup_{n\in\mathbb{N}}\overline{nV} = Y$ would have an empty interior, a contradiction (since set Y itself is open and nonempty in Banach space Y). So there is some $n \in \mathbb{N}$ with $\overline{nV} = n\overline{V}$ having a nonempty interior, say $ny = w$ is an interior point of $n\overline{V}$. That is, y is an interior point of $\overline{V}$, so for some $s > 0$ we have that $y + B'(s) \subseteq \overline{V}$.

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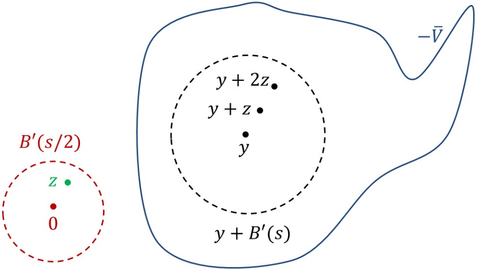
Theorem 3.5 (continued 2)

Proof (continued). Now suppose $z \in B'(s/2)$. Then $-(y+z) \in -\overline{V} = \overline{-V} = \overline{V}$ (by Lemma 3.4(b)). Also, $y + 2z \in \overline{V}$:



So

$$\begin{aligned} z &= (y+2z) - (y+z) \in \overline{V} + \overline{V} \subseteq \overline{V+V} \text{ by Lemma 3.4(c)} \\ &= \overline{T(B(1/2)) + T(B(1/2))} = \overline{T(B(1/2) + B(1/2))} = \overline{T(B(1))}. \end{aligned}$$

So with $r = s/2$ we have $B'(s/2) = B'(r) \subseteq \overline{T(B(1))}$.

Theorem 3.5 (continued 3)

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Given a surjective (onto) $T \in \mathcal{B}(X, Y)$ where X and Y are Banach spaces, if $U \subseteq X$ is open then $T(U)$ is open.

Proof (continued). STEP 3. From Step 2, we have $B'(r) \subseteq \overline{T(B(1))}$ for some $r > 0$ (namely, $r = s/2$). For any nonnegative $k \in \mathbb{Z}$ we have $2^{-k}B'(r) \subseteq 2^{-k}\overline{T(B(1))}$ and so by Lemma 3.4(d) $B'(r/2^k) \subseteq \overline{T(B(1/2^k))}$. So for any $z \in B'(r/2^k)$, either $z \in T(B(1/2^k))$ or z is a limit point of $T(B(1/2^k))$ (by Theorem 2.2.A). So for any $\varepsilon > 0$ there exists $x \in B(1/2^k)$ such that $\|z - Tx\| < \varepsilon$.

Suppose $y \in B'(r)$. Choose $x_1 \in B(1)$ such that $\|y - Tx_1\| < r/2$ (from above with $k = 0$ and $\varepsilon = r/2$). So $y - Tx_1 \in B'(r/2)$. Then choose $x_2 \in B(1/2)$ such that $\|(y - Tx_1) - Tx_2\| < r/4$ (from above with $k = 1$ and $\varepsilon = r/4$).

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Theorem 3.5 (continued 4)

Proof (continued). STEP 3 (continued). Then inductively choose $x_n \in B(1/2^{n-1})$ such that

$$\|(y - Tx_1 - Tx_2 - \cdots - Tx_{n-1}) - Tx_n\| = \|y - T(x_1 + x_2 + \cdots + x_n)\| < r/2^n \ (*)$$

(from above since $y - T(x_1 + x_2 + \cdots + x_n) \in B'(1/2^{n-2})$ with $k = n - 1$ and $\varepsilon = r/2^n$). Now $\|x_n\| < 1/2^{n-1}$, so $\sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} 1/2^{n-1} = 2$ and $\sum_{n=1}^{\infty} x_n$ is absolutely convergent and therefore convergent by Theorem 2.12, say $x = \sum_{n=1}^{\infty} x_n$. By the Triangle Inequality, $\|x\| \le \sum_{n=1}^{\infty} \|x_n\| < 2$. Now

$$\begin{aligned} Tx &= T\left(\sum_{n=1}^{\infty} x_n\right) \\ &= \sum_{n=1}^{\infty} Tx_n \text{ since } T \text{ is bounded and so continuous by Theorem 2.6} \\ &= y \text{ by } (*). \end{aligned}$$

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Proof (continued). STEP 3 (continued). So $x \in B(0; 2) = B(2)$ and $y = Tx \in T(B(2))$. Since y is an arbitrary element of $B'(r)$ then $B'(r) \subseteq T(B(2))$, and so $\frac{1}{2}B'(r) \subseteq \frac{1}{2}T(B(2))$ or $B'(r/2) \subseteq T(B(1))$ by Lemma 3.4(d). With $\delta = r/2$, we have shown that $B'(\delta) \subseteq T(B(1))$ and the result now holds by STEP 1. □